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THE THEORY OF LINEAR VECTOR FUNCTIONS.

By CHARLES J. JOLY, M.A., Fellow of Trinity College, Dublin.

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XVI.

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[Read December 10, 1894.]

ART.		PAGE
1.	Relations between the axes of a linear vector function ϕ and of its conjugate ϕ'. Supplemental triangles on a sphere,	597
2.	Case of equal roots. Invariants of ϕ in terms of its rotation vector and of the invariants of its self-conjugate part,	599
3.	Congruency of lines parallel to the axes drawn through the extremities of varying rotation vectors. Adjoint congruency. Pedal and nodal surfaces. Root quadrics and sphero-conics. Singular planes of the congruency,	601
4.	Focal and discriminating surface,	603
5.	Metrical properties of focal surface deduced by the aid of a semi-cubical parabola,	606
6.	Detailed description of its plane sections,	609
7.	Intersections of root quadrics with the focal surface. Singular tangent planes,	612
8.	Metrical properties of the pedal and nodal surfaces,	614
9.	Construction of the congruency by means of a quadric. Lines in central planes,	617
10.	Lines in planes through an arbitrary point. Quintic surface generated by lines meeting an arbitrary line. Ordinary singularities are a triple line and a nodal ellipse,	619
11.	Director cone of quintics,	621
12.	Form of quintics,	623
13.	System of director cones of quintics whose triple lines pass through a fixed point,	625
14.	Locus of double edges,	627
15.	Geometrical relations between the corresponding director cones of the two congruencies,	628
16.	Equations of quintic surface. Extraordinary singularities. Cylindroids,	629
17.	Lines of the congruencies parallel to fixed planes generate cylindroids,	631
18.	Nodal ellipses and cylindroids. Envelope of planes containing lines which meet at a given angle. Locus of the points of intersection of these lines. Special cases are the focal and central nodal surfaces,	634
19.	Nodal ellipses and nodal surfaces,	636
20.	Plane sections of quintics,	637
21.	Homographic systems used in discussing plane cubics. Homographic generation of quintics,	640
22.	Torsal generators of quintics,	642
23.	Reciprocal of a quintic,	*ib.*
24.	Quintic generated by a line meeting fixed curves. Hessian of a quintic,	644
25.	Congruency in relation to the theory of screws. "Screw complex" of the third order,	645

1. It is proved in treatises on Quaternions that a linear vector function ϕ satisfies a symbolical cubic equation

$$f = \phi^3 - m_1\phi^2 + m_2\phi - m_3 = (\phi - c_1)(\phi - c_2)(\phi - c_3) = 0,$$

c_1, c_2, and c_3 being the roots of $c^3 - m_1c^2 + m_2c - m_3 = 0$; in other words, the result of operating by f on any vector is zero.

The axes of ϕ are the vectors ρ_1, ρ_2, and ρ_3, satisfying $(\phi - c_1)\,\rho_1 = 0$; $(\phi - c_2)\rho_2 = 0$; $(\phi - c_3)\rho_3 = 0$; and these solutions are unique.

If ρ is any vector,

$$(\phi - c_2)\,(\phi - c_3)\,\rho \parallel \rho_1; \quad (\phi - c_3)\,(\phi - c_1)\,\rho \parallel \rho_2; \quad (\phi - c_1)\,(\phi - c_2) \parallel \rho_3,$$

since $f\rho = 0$, and $(\phi - c_1)\,\rho_1 = 0$, and the solutions are unique. If, however, ρ is coplanar with ρ_2 and ρ_3, $(\phi - c_2)(\phi - c_3)\,\rho = 0$. Thus $(\phi - c_1)\,\rho$ is coplanar with ρ_2 and ρ_3.

A real linear vector function has always one real axis. One root c_1 of the cubic in c is real, and if c_2 and c_3 are imaginary, $(\phi - c_2)\,(\phi - c_3)$ is real, and therefore the result of operating by this on a real vector ρ is real. On the other hand $(\phi - c_3)\,(\phi - c_1)$ and $(\phi - c_1)(\phi - c_2)$ are imaginary, and the axes ρ_2 and ρ_3 are parallel to imaginary vectors.

If $\phi = \Sigma\tau S\tau'$, the conjugate of ϕ is $\phi' = \Sigma\tau' S\tau$. For any vectors ρ and ρ', $S\rho'\phi\rho = S\rho\phi'\rho'$; $S\rho'\phi^2\rho = S\rho\phi'^2\rho'$, &c. In particular, $S\rho' f\rho = S\rho f'\rho'$. But $f\rho = 0$, so $f'\rho' = 0$ also, ρ and ρ' being quite arbitrary. Hence ϕ' satisfies the symbolical equation,

$$f' = \phi'^3 - m_1\phi'^2 + m_2\phi' - m_3 = (\phi' - c_1)\,(\phi' - c_2)\,(\phi' - c_3) = 0,$$

and ϕ' has the same roots as ϕ.

The axes of ϕ' are in general different from those of ϕ, being given by

$$(\phi' - c_2)\,(\phi' - c_3)\rho \parallel \rho'_1; \quad (\phi' - c_3)\,(\phi' - c_2)\,\rho \parallel \rho'_2; \quad (\phi' - c_1)\,(\phi' - c_2)\rho \parallel \rho'_3.$$

If ρ is any vector, $(\phi' - c_1)\rho$ is at right angles to ρ_1. For

$$S\rho_1(\phi' - c_1)\,\rho = S\rho(\phi - c_1)\rho_1 = 0.$$

But $(\phi' - c_1)\,\rho$ is the general expression for a vector coplanar with ρ'_2 and ρ'_3. ρ_1 is therefore at right angles to ρ'_2 and ρ'_3.

Hence, if $\rho_1\rho_2\rho_3$ and $\rho'_1\rho'_2\rho'_3$ are unit coinitial vectors, their extremities determine supplemental triangles on a unit sphere. The axes of ϕ' in terms of those of ϕ are

$$U\rho'_1 = UV\rho_2\rho_3; \quad U\rho'_2 = UV\rho_3\rho_1; \quad U\rho'_3 = UV\rho_1\rho_2,$$

also

$$U\rho_1 = UV\rho'_2\rho'_3; \quad U\rho_2 = UV\rho'_3\rho'_1; \quad U\rho_3 = UV\rho'_1\rho'_2.$$

If a more formal proof of this be desired, it is sufficient to remark that $S\rho_1\sigma = 0$—the condition that σ should be at right angles to ρ_1—may be written $S(\phi - c_2)(\phi - c_3)\rho\sigma = 0$, or $S\rho(\phi' - c_2)(\phi' - c_3)\sigma = 0$, and as ρ is arbitrary $(\phi' - c_2)(\phi' - c_3)\sigma = 0$. Or, again, as $(\phi - c_2)\rho$ is at right angles to ρ'_2, $(\phi - c_2)(\phi - c_3)\rho$ is at right angles to ρ'_2 and to ρ'_3.

If any equation, such as $(\phi'^2 + x\,\phi' + y)\,\rho = 0$ be satisfied by two values of ρ, $(\phi^2 + x\,\phi + y)\sigma$ is *the* vector at right angles to the two vectors satisfying the equation. But σ is quite arbitrary, and this vector is unique. Hence $\phi^2 + x\,\phi + y$ must be one of the three expressions $(\phi - c_2)(\phi - c_3)$, $(\phi - c_3)(\phi - c_1)$, $(\phi - c_1)(\phi - c_2)$ as follows at once, by writing $\sigma = p\rho_1 + q\rho_2 + r\rho_3$ where pqr are arbitrary scalars.

2. Since $S\rho\phi\rho = S\rho\phi'\rho$, or $S\rho(\phi - \phi')\rho = 0$, where ρ is arbitrary, $(\phi - \phi')\rho$ is at right angles to ρ, and therefore $(\phi - \phi')\rho = 2V\epsilon\rho$. ϵ is the rotation vector of ϕ, or the non-conjugate part; $\phi + \phi' = 2\Phi$ is self-conjugate; $-\epsilon$ is the rotation vector of ϕ'. The vector functions ϕ and ϕ' may therefore be written

$$\phi = \Phi + V\epsilon, \quad \phi' = \Phi - V\epsilon.$$

ϵ is the only real solution of

$$(\phi - \phi')\rho = 0.$$

Since

$$(\phi - c_1)\epsilon = (\phi' - c_1)\epsilon = (\Phi - c_1)\epsilon,$$

$(\Phi - c_1)\epsilon$ is at right angles to ρ_1 and ρ'_1, or parallel to $V\rho_1\rho'_1$. The vectors ϵ, $\Phi\epsilon$, $V\rho_1\rho'_1$, $V\rho_2\rho'_2$, $V\rho_3\rho'_3$ are coplanar, and the normal to their common plane is $V\epsilon\Phi\epsilon$. Thus, $SV\epsilon\Phi\epsilon V\rho_1\rho'_1 = 0$, that is $V\epsilon\Phi\epsilon$ is coplanar with ρ_1 and ρ'_1, and pierces the unit sphere in the common orthocentre of the triangles $\rho_1\rho_2\rho_3$ and $\rho'_1\rho'_2\rho'_3$.

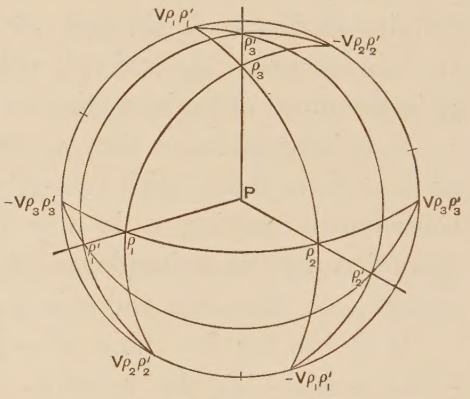

FIG. 1.

Orthographic projection on a plane at right angles to $V\epsilon\phi\epsilon$. P is the projection of $UV\epsilon\phi\epsilon$.

If ϵ is coplanar with ρ_2 and ρ_3, $2S\epsilon\rho_2\rho_3 = S(\phi - \phi')\rho_2\rho_3 = (c_2 - c_3)S\rho_2\rho_3$ vanishes, and either ρ_2 and ρ_3 are at right angles or the root $c_2 = c_3$.

If ϵ coincides with ρ_1, $(c_3 - c_1) S\rho_3\rho_1 = (c_1 - c_2) S\rho_1\rho_2 = 0$; and if the roots are not equal, ρ_1 is at right angles to ρ_2 and ρ_3 and coincides with ρ'_1. If ρ_1 coincides with ρ'_1, $V\epsilon\rho_1 = 0$. In the case of equality of roots ($c_2 = c_3$), $S\epsilon\rho_2\rho_3 = 0$, and either ρ_2 is coincident with ρ_3 or ϵ coplanar with ρ_2 and ρ_3.

As the roots c_2 and c_3 tend towards coincidence for varying values of ϵ, $(\Phi - c_2)\epsilon$ and $(\Phi - c_3)\epsilon$ approach likewise. Provided ρ_1 does not become indefinitely close to $(\Phi - c_2)\epsilon$ or $(\Phi - c_3)\epsilon$, the planes through $\rho_1\rho_2$ and through $\rho_1\rho_3$ close together. In this case ρ_2 and ρ_3 will ultimately coincide, but the plane through them continues to be definite as it passes through $(\Phi - c_1)\epsilon$. In like manner ρ'_2 and ρ'_3 coincide. ρ_2 is therefore at right angles to ρ'_2.

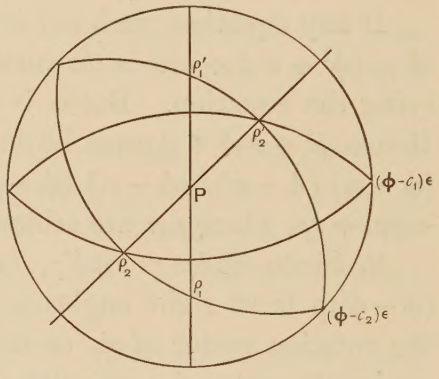

FIG. 2.

Orthographic projection on a plane at right angles to $V\epsilon\phi\epsilon$ (marked P), when $c_2 = c_3$.

It is here assumed that ρ_1 does not tend to coincidence with $(\Phi - c_2)\epsilon$, or $(\Phi - c_3)\epsilon$. But keeping $c_2 = c_3$, and consistently varying ϵ, it may be inferred that there is in general no exception to the coincidence of ρ_2 and ρ_3, provided $(\Phi - c_2)\epsilon$ does not vanish. In this case there is an exception.

If Φ has for its cubic $\Phi^3 - M_1\Phi^2 + M_2\Phi - M_3 = 0$, and if $M(c) = M_3 - M_2 c + M_1 c^2 - c^3$, the relations between the coefficients of the cubics ϕ and Φ are as follows:—

If $\phi\rho_1 = c_1\rho_1 = \Phi\rho_1 + V\epsilon\rho_1$, $\rho_1 = -(\Phi - c_1)^{-1} V\epsilon\rho_1$, or provided $M(c_1)$ is not zero, $\rho_1 = -\dfrac{1}{M(c_1)} V(\Phi - c_1)\epsilon\,(\Phi - c_1)\rho_3 = \dfrac{1}{M(c_1)} V(\Phi - c_1)\epsilon\; V\epsilon\rho_1$.

Now,

$$V(\Phi - c_1)\epsilon\; V\epsilon\rho_1 = \rho_1 S\epsilon(\Phi - c_1)\epsilon - \epsilon S\rho_1(\Phi - c_1)\epsilon.$$

So $S\rho_1(\Phi - c_1)\epsilon = 0$, and $M(c_1) = S\epsilon(\Phi - c_1)\epsilon$. Consequently, the cubic of ϕ is $M(c) - S\epsilon(\Phi - c)\epsilon = 0$, as this is satisfied by c_1, c_2 and c_3. Hence $m_1 = M_1$; $m_2 = M_2 - \epsilon^2$; $m_3 = M_3 - S\epsilon\Phi\epsilon$.

3. If $\Phi\rho_1 + V\epsilon\rho_1 = c_1\rho_1$, $c_1 = S\rho_1^{-1}\Phi\rho_1$ and, if x is an arbitrary scalar, $\epsilon = V\rho_1^{-1}\Phi\rho_1 + x\rho_1$. This is the equation of a right line parallel to ρ_1, and the linear vector function whose self-conjugate part is Φ, and whose rotation vector is the vector from the origin to any point on this line, has a root equal to c_1, and an axis parallel to ρ_1. It will be observed that, given the direction of ρ_1, the line is uniquely determined as well as the root c_1. If $\rho = V\alpha^{-1}\Phi\alpha + x\alpha$, and α is arbitrarily varied, the system of lines constitutes a congruency. The congruency is of the third order, since through any point ρ pass three lines parallel to the axes of $\Phi + V\rho$. The congruency is of the second class, since in any plane $S\lambda\rho + 1 = 0$ lie two lines, α being determined by the intersection of the quadric cone, $S\lambda\alpha^{-1}\Phi\alpha + 1 = 0$, with the plane $S\lambda\alpha = 0$.

If $\alpha' = V\alpha^{-1}\Phi\alpha$, $V\alpha'^{-1}\Phi\alpha' = \dfrac{V.\, V\alpha\,\Phi\alpha\,\Phi\, V\alpha\,\Phi\alpha}{V^2\alpha\,\Phi\alpha}$

$$= M_3\,\frac{V.\, V\alpha\,\Phi\alpha\, V\Phi^{-1}\alpha\alpha}{V^2\alpha\,\Phi\alpha} = \alpha\,\frac{S\alpha\,\Phi\alpha\,\Phi^2\alpha}{V^2\alpha\Phi\alpha},$$

and if the tensor of α is chosen to satisfy $S\alpha\,\Phi\alpha\,\Phi^2\alpha = V^2\alpha\,\Phi\alpha$, the simple reciprocal relation $V\alpha'^{-1}\Phi\alpha' = \alpha$ holds. Thus, $\rho = \alpha + y\alpha'$ is the second line of the congruency coplanar with $\rho = \alpha' + x\alpha$ and the origin, and these lines cut at right angles.

The quartic surface $S\rho\Phi\rho\Phi^2\rho = V^2\rho\Phi\rho$ is the locus of the feet of central perpendiculars on lines of the congruency, and may be conveniently designated the pedal surface.

The vector to the point of intersection of these two lines is $\rho = \alpha + \alpha'$, and two of the axes of the linear vector function, of which this is the rotation vector, are coplanar with it. If therefore a and a' are the roots,

$$(\Phi + V\rho - a)(\Phi + V\rho - a')\rho = 0\,;\ \text{or,}\ \Phi^2\rho + V\rho\Phi\rho - (a+a')\Phi\rho + aa'\rho = 0.$$

And, on elimination of a and a', the point of intersection of the lines is found to lie on the surface (which may be called the nodal surface) $S\rho\Phi\rho\Phi^2\rho = -V^2\rho\Phi\rho$. This may be obtained from the pedal surface by *reversing* the vectors to points on it, viz. by simply changing the sign of ρ.

In addition to the congruency $\rho = V\alpha^{-1}\Phi\alpha + x\alpha$, the adjoint congruency $\rho = -V\alpha^{-1}\Phi\alpha + x\alpha$ may be considered. The pedal surface of one is the

nodal surface of the other. The lines of the second congruency, through any point ϵ, are parallel to the axes of $\Phi - V\epsilon$. Thus, by what has been proved in Art. 1, a line of one congruency cuts, at right angles, those of the other which it meets, whose roots are not equal to its own.

The congruencies consist of the generators of the system of concyclic quadrics, $S\rho(\Phi - a)\rho = M(a)$. If ρ is any point on a line whose root is a, this equation is satisfied, being simply the root cubic of the function $\Phi + V\rho$. The generators of one system belong to one congruency, and those of the opposite system to the other, otherwise a linear vector function $\Phi + V\rho$ would have equal roots and two distinct axes corresponding to these equal roots, contrary to Art. 2.

The discriminant of a quadric of the system is $(M(a))^2$. When this is zero, the quadric breaks up into a pair of cyclic planes $S\rho(\Phi - c)\rho = 0$. These are singular planes of the congruency, and contain an infinite number of lines.

These quadrics have the points $\rho^2 = S\rho\Phi\rho = 0$ in the plane infinity common, each of these four points being counted twice.

Two quadrics intersect in the sphero-conic

$$\rho^2 = M_2 - M_1(a_1 + a_2) + a_1^2 + a_1a_2 + a_2^2; \quad S\rho\Phi\rho = M_3 - M_1a_1a_2 + a_1a_2(a_1 + a_2);$$

or, if $M_1 = a_1 + a_2 + a_3$, the sphero-conic is

$$\rho^2 = M_2 - (a_2a_3 + a_3a_1 + a_1a_2); \quad S\rho\Phi\rho = M_3 - a_1a_2a_3.$$

In this form it is evident that a third quadric passes through the sphero-conic. Vector functions defined by rotation vectors to points on this sphero-conic have their roots equal to a_1, a_2, and a_3.

It is well to point out in this place, that if c_1, c_2, and c_3 are the roots of Φ, and i, j, and k its axes, that the root quadric a is

$$(a - c_1)x^2 + (a - c_2)y^2 + (a - c_3)z^2 = (c_1 - a)(c_2 - a)(c_3 - a).$$

This is real only if $c_1 > a > c_3$, where $c_1 > c_2 > c_3$, and when real it is necessarily a hyperboloid of one sheet. For suppose in turn $c_1 > a > c_2$ and $c_2 > a > c_3$, and the nature of the quadric is evident.

4. It is known that lines of a congruency are bitangents to a certain surface—the focal surface. If A and A' are the two points of contact of a line of the congruency, the plane containing this and the consecutive line which passes through A, touches the focal surface at A'.

Suppose a line $t\epsilon$ drawn through the origin. The roots of the function $\Phi + tV\epsilon$ are given by $t^2 S\epsilon(\Phi - c)\epsilon = M(c)$. When $t = 0$, these roots are real if Φ is real and the axes are distinct. As t increases, two of the roots coincide, and on further increase of t, become imaginary. Two of the axes, in like manner, coincide and become imaginary. The point $t\epsilon$ is a point A on the focal surface, if two of the lines of the congruency through it coincide.

The equation of the focal surface is therefore obtainable by putting $a_1 = a_2 = a$ in the equations of the sphero-conic in the last article, and eliminating a. The focal surface is the locus of the sphero-conics

$$\rho^2 = M_2 - 2aM_1 + 3a^2, \quad S\rho\Phi\rho = M_3 - a^2 M_1 + 2a^3;$$

or, what is equivalent, it is the envelope of the system of root quadrics, or its equation is the discriminant of the root cubic:—

$$F = \frac{4}{27}(M_2 - \frac{1}{3}M_1^2 - \rho^2)^3 + \left[M_3 - \frac{1}{3}M_1 M_2 + \frac{2}{27}M_1^3 - S\rho(\Phi - \frac{1}{3}M_1)\rho\right]^2 = 0.$$

Through any point within this surface three real lines pass, and through any point outside it, only one. For inside ($F < 0$) three roots are real, and outside ($F > 0$) two roots are imaginary, and in Art. 1 the reality of the lines is deduced from the reality of the roots.

Again, the reciprocal of the focal surface may be thus obtained. If $S\lambda\rho + 1 = 0$ is the plane touching the quadric $S\rho(\Phi - a)\rho = M(a)$ at ϖ, $\lambda M(a) = -(\Phi - a)\varpi$, on equating the equation of the plane to $S\rho(\Phi - a)\varpi = M(a)$. Hence $M(a)S\lambda(\Phi - a)^{-1}\lambda = 1$, or,

$$\text{(since } M(a) = (\Phi - a)[\Phi^2 + a\Phi + a^2 - M_1(\Phi + a) + M_2]),$$
$$M_3 S\lambda\Phi^{-1}\lambda - 1 + aS\lambda(\Phi - M_1)\lambda + a^2\lambda^2 = 0,$$

is the tangential equation of the quadric. Finding the envelope of this system as a varies, $4\lambda^2(M_3 S\lambda\Phi^{-1}\lambda - 1) - S^2\lambda(\Phi - M_1)\lambda = 0$ is the reciprocal of the focal surface.

This may be considered anew. The tangential equation of the quadric

is a quadratic in a. Consequently, an arbitrary plane touches two root quadrics, and contains two lines of each congruency, whose roots are given by the quadratic. When the roots are equal, the lines of each pair coincide, and the plane touches the focal surfaces of both congruencies. Both congruencies have the same focal surface, as may be seen in other ways. One proof is afforded by the form of the pointal equation of the focal surface, which shows it to be *central*. If the line $\alpha' + x\alpha$ touches it, the line $-\alpha' + x\alpha$ touches it also. Another proof follows from the fact that the adjoint congruency may be derived from the original by change of sign of Φ, and therefore of M_1, M_3, and of any root.

Through any point on a line of one congruency pass two lines of the other, at right angles to it. The lines of shortest distance between two lines of one congruency, are lines of the other. This converse is included in a more general property. Every line of the first congruency, at right angles to a given direction β, meets a fixed line of the second congruency. For operating by $S\beta$ on $\Phi\alpha + V\epsilon\alpha = a\alpha$, $S\alpha(\Phi\beta - V\epsilon\beta) = 0$, and this is satisfied if $\epsilon = -V\beta^{-1}\Phi\beta + y\beta$.

In particular, the shortest distances between a line of the first congruency and a second consecutive to it are lines of the adjoint congruency. Through a point on the line inside the focal surface, two lines of shortest distance pass, and these two coincide at what are called the extreme points (B and B') on the given line. Outside these points, by the general theory of congruencies, the line is met by no real lines of shortest distance. This shows that B and B' are the remaining points at which the (bitangent) line meets the sextic focal surface.

For an arbitrary congruency the locus of the extreme points (BB') is a surface distinct from the locus of the focal points (AA'). In this congruency these two surfaces are the same. The points BB' are focal points, and AA' extreme points on the lines of the adjoint congruency. This is a general property connecting a congruency and the congruency formed by lines of shortest distance between its consecutive lines.

The roots of the lines through $\alpha' + t\alpha$, ($\alpha' = V\alpha^{-1}\Phi\alpha$) are a, and those given by the quadratic $g^2 + ag + a^2 - M_1(g + a) + M_2 - \alpha'^2 - t^2\alpha^2 = 0$, and are the same for equal and opposite values of t.

If a second root is equal to a, t is given by $t_1^2 T^2\alpha = -3a^2 + 2aM_1 - M_2 + \alpha'^2 = (c-a)(a-b)$ if a, b, and c are the three roots of the lines through α'. This determines the focal points. If the roots of the quadratic in g are equal, they are equal to $\frac{1}{2}(M_1 - a)$; and if t_2 is the corresponding value of t, $t_2^2 T^2\alpha = \frac{1}{4}(b-c)^2$; and this determines the extreme points on the line. $(t_2^2 - t_1^2)T^2\alpha = \frac{1}{4}(b+c-2a)^2$ is always positive, or the extreme points lie outside the focal points.

The values derived for t_2 are always real. On the other hand t_1 may be imaginary. The points coincide if $b+c-2a=0$, or if $a = \frac{1}{3}M_1$. The corresponding line touches the surface twice in three consecutive points. If $a = b$ or c, the line is an undulatory tangent to the surface, and if $a = b = c = \frac{1}{3}M_1$, it meets it in six consecutive points. These points are to be determined by the intersection of the sphero-conic $S\rho\Phi\rho = M_3 - \frac{1}{27}M_1^3$, $\rho^2 = M_2 - \frac{1}{3}M_1^2$ with the pedal surface. Since in the determination of the points AA' and BB' the equations in t have equal and opposite roots, the foot of the central perpendicular on the line is the common centre of both pairs of points.

The reality of the focal points may be determined in a more useful form. The double root at the focal points being a, the single root is $M_1 - 2a$. This, if the point is real, must satisfy $c_1 > M_1 - 2a > c_3$, or $\frac{1}{2}(c_1 + c_2) > a > \frac{1}{2}(c_2 + c_3)$, which is more stringent than the inequality $c_1 > a > c_3$. The double root at the extreme points is $\frac{1}{2}(M_1 - a)$, and leads to the inequality $c_1 + (c_2 - c_3) > a > c_3 - (c_1 - c_2)$, which is always satisfied if $c_1 > c_2 > c_3$, these being the roots of Φ.

The quadratic giving the roots of the lines through any point on $\alpha' + t\alpha$, may be reduced to $(g-b)(g-c) + t^2 T^2\alpha = 0$, and the roots of this are $g = \frac{1}{2}(b+c) \pm \frac{1}{2}(b-c)\sqrt{\left(1 - \frac{t^2}{t_2^2}\right)}$, and are real if t is less than t_2; they separate the roots b and c, and *a fortiori* satisfy the inequality. The portion of the line intercepted between the extreme points lies within the focal surface. Hence the pedal surface cannot cut, though it may touch, the focal surface in real points. For to a real point α' on the pedal surface corresponds the real line $\alpha' + tV\alpha'^{-1}\Phi\alpha'$.

The line $\alpha' + y\alpha$ passes through $\alpha + \alpha'$. The roots at this point are

a, a', and a'', or $M_1 - a - a'$. Hence $T^2\alpha = (b-a')(a'-c)$ and $(t_1^2-1)T^2\alpha = (a-a')(a''-a)$, and $(t_2^2-1)T^2\alpha = \frac{1}{4}(2a'-b-c)^2 = \frac{1}{4}(a'-a'')^2$. From symmetry for the line $\alpha + y\alpha'$, the focal points are given by

$$(t_1'^2 - 1)\,T^2\alpha' = (a' - a'')(a - a') \text{ and } (t_2'^2 - 1)\,T^2\alpha' = \tfrac{1}{4}(a - a'')^2.$$

5. The congruency $\rho = V\alpha^{-1}\Phi\alpha + x\alpha$ evidently depends on the differences of the roots of Φ, and calculation is generally shorter when Φ is replaced by $\Psi + \frac{1}{3}M_1$. The cubic of Ψ is

$$\Psi^3 + M_2\Psi - M_3 = (\Psi - G_1)(\Psi - G_2)(\Psi - G_3) = 0,$$

the symbols M_2 and M_3 here used being, however, not the same as those previously employed. I suppose the roots of Ψ to satisfy the inequality $G_1 > G_2 > 0 > G_3$. This imposes no restriction on the generality of the function. The focal surface is now the locus of the sphero-conics $\rho^2 = M_2 + 3g^2$; $S\rho\Psi\rho = M_3 + 2g^3$, and its form may be conveniently discussed by the aid of the semi-cubical parabola

$$x + M_2 + 3g^2 = 0, \quad y + M_3 + 2g^3 = 0.$$

To x = constant, corresponds a sphere, to $y = g'x$ a quadric cone of a concyclic system, to $y - g'x + M(g') = 0$ a root quadric, and to a point on the curve a sphero-conic on the surface.

The cubic has a cusp at $x = -M_2$, $y = -M_3$. If $x = 0$, $y = -M_3 \pm \sqrt{-\frac{4}{27}M_2^3}$, and since $4M_2^3 + 27M_3^2$ is negative, one value of y is negative. The origin lies between the branches of the curve, as in the figure, and as $-M_2$ and $-M_3$ are both positive, the cusp Q is as drawn. (Fig. 3.)

The sum of the reciprocals of the parameters g, in which any line meets the cubic, is zero. The lines corresponding to the root quadrics are tangents, the parameters of the points of intersection of the line g' being given by $2g^3 - 3g'g^2 + g'^3 = 0$, or $(2g + g')(g - g')^2 = 0$. The tangents from the origin are $y = G_1x$, $y = G_2x$, $y = G_3x$, and answer to the pairs of cyclic planes, or, regarded as belonging to the system of concyclic cones, $S\rho(\Psi - g)\rho = 0$, the first and third are (real) cones of zero vertical angle, and the second is a pair of real planes.

For reality of the cones, $G_1 >= g >= G_3$; and for reality of the spheres x must be positive. Hence the portion of the curve $T'_1QT'_3$ corresponds to the real part of the focal surface, and the rest of the curve corresponds to the imaginary part. T'_1OT_1 is the tangent $y = G_1x$; $OT'_2T'_2$ is $y = G_2x$

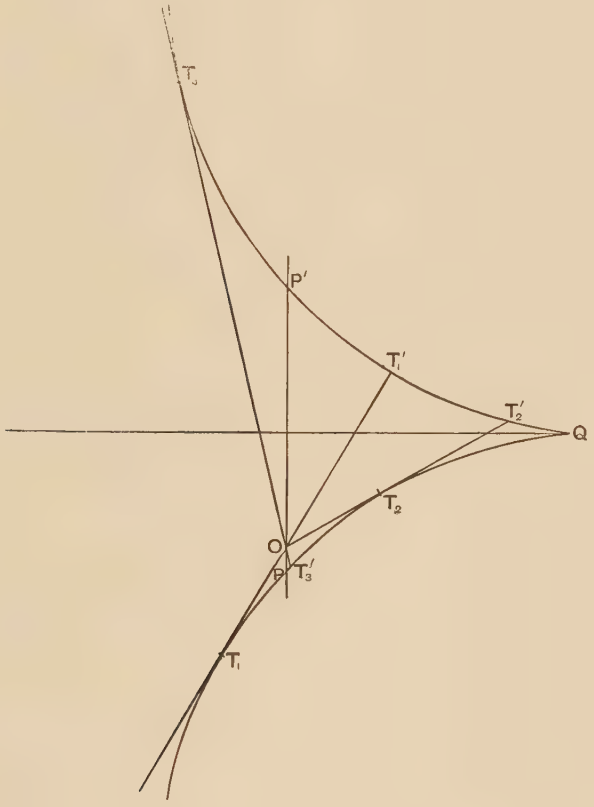

FIG. 3.

and T'_3OT_3, $y = G_3x$. The abscissæ of the points of contact are $x_1 = r_1^2 = (G_3 - G_1)(G_1 - G_2)$; $x_2 = r_2^2 = (G_1 - G_2)(G_2 - G_3)$; $x_3 = r_3^2 = (G_2 - G_3)(G_3 - G_1)$; and those of the tangentials are

$$x'_1 = r'_1{}^2 = \tfrac{1}{4}(G_2 - G_3)^2; \quad x'_2 = r'_2{}^2 = \tfrac{1}{4}(G_3 - G_1)^2; \quad x'_3 = r'_3{}^2 = \tfrac{1}{4}(G_1 - G_2)^2.$$

The radii of the corresponding spheres are denoted by r_1, &c. For any

order of magnitude two of the quantities x_1, x_2, x_3 are negative, and the third positive. The cyclic planes touch the focal surface along a circle (r_1, r_2, or r_3), and cut it along another circle (r'_1, r'_2, or r'_3). The real part of the focal surface is bounded externally by the sphere whose radius is $\sqrt{-M_2} = r_0$, and internally by the sphere whose radius is the smaller of the two values r'_1, r'_3, viz. r'_3.

Spheres whose radii satisfy $r'_1 > r > r'_3$ cut the surface in one real sphero-conic; and if $r_0 > r > r'_1$ in two real sphero-conics.

Cones of the concyclic system $S\rho(\Psi - g)\rho = 0$ cut the surface in three real sphero-conics if $G_2 > g > \frac{M_3}{M_2}$, and otherwise in one real sphero-conic, provided the cone is real.

These properties are evident on inspection of the cubic curve. $y = \frac{M_3}{M_2}x$ is the equation of the line OQ. Thus, central lines lying outside the cone $M_2 S\rho\Psi\rho - M_3\rho^2 = 0$, or $S\rho\Psi^3\rho = 0$, and within its real cyclic planes meet the focal surface in six real points. The line OQ meets the cubic again in a point whose abscissa is $x' = r'^2 = -\frac{4M_2^3 + 27M_3^2}{4M_2^2}$, and if a central line meet the surface in six real points, they lie outside the sphere whose radius is r', or, more definitely, two pairs of the six points lie outside the sphere whose radius is r_2, and the other pair lies between the spheres r' and r_2.

The tangent $y - gx + M(g) = 0$ meets the cubic again in a point whose abscissa is greater than that of its point of contact. If the point of contact corresponds to a real sphero-conic $-\frac{1}{2}G_3 > g > -\frac{1}{2}G_1$, the parameter increasing from $-\frac{1}{2}G_1$ at T'_1 to zero at Q, and thence to $-\frac{1}{2}G_3$ at T'_3. Hence the condition of real bitangency of a line of the congruency with the focal surface.

Every line through the cusp meets the cubic in two consecutive points. Every quadric through the sphero-conic $S\rho\Psi\rho = M_3$, $\rho^2 = M_2$, meets the focal surface in consecutive sphero-conics. This sphero-conic is cuspidal, and $S\rho\Psi\rho = M_3$ is enveloped by the cuspidal tangent planes. To the inflexion at infinity on the cubic answers the imaginary cuspidal circle at infinity on the surface. The asymptote of the cubic is parallel to x = const.

This line passes through the inflexion as the corresponding sphere passes through the circle at infinity.

These deductions are, however, more obvious on inspection of the equation of the focal surface, viz. $4(M_2-\rho^2)^3+27(M_3-S\rho\Psi\rho)^2=0$.

6. If ρ and $\rho+d\rho$ are the vectors to two neighbouring points on the focal surface, $\rho^2=M_2+3g^2$, $S\rho\Psi\rho=M_3+2g^3$, and $(\rho+d\rho)^2=M_2+3(g+dg)^2$, $S(\rho+d\rho)\Psi(\rho+d\rho)=M_3+2(g+dg)^3$. If the points are very close, $S\rho d\rho=3gdg$, and $S\rho\Psi d\rho=3g^2dg$, and eliminating dg, $Sd\rho(\Psi-g)\rho=0$, so $S(\varpi-\rho)(\Psi-g)\rho=0$, is the equation of the tangent plane at ρ. If ρ is a double point on the surface the equation vanishes. $(\Psi-g)\rho$ must vanish, or g must be G_1, G_2, or G_3, and ρ must be parallel to i, j, or k (the axes of Ψ). Referring to the last article for the radii of circles of contact of the cyclic planes with the surface, $\pm r_1 i$, $\pm r_2 j$, $\pm r_3 k$ lie on the sphero-conics whose parameters are G_1, G_2, and G_3, respectively, and are therefore double points on the surface. The points $\pm r_2 j$ alone are real.

Retaining terms of the second order in $d\rho$ and dg,

$$2S\rho d\rho+(d\rho)^2=6gdg+3(dg)^2.$$

$$2S\Psi\rho d\rho+Sd\rho\Psi d\rho=6g^2dg+6g(dg)^2,$$

and on squaring the first of these $S^2\rho d\rho=9g^2dg^2$, so on elimination of dg, $2S(\Psi-g)\rho d\rho+Sd\rho(\Psi-g)d\rho=\frac{1}{3g}S^2\rho d\rho$. This, if $d\rho$ be regarded as variable, is the equation of a quadric having contact of the second order with the focal surface at ρ. At the points, $\pm r_2 j$, the quadric is $Sd\rho(\Psi-G_2)d\rho=\frac{r_2^2}{3G_2}S^2jd\rho$, a real tangent cone touched by the cyclic planes. The tangent cones at the other double points are essentially imaginary. In Cartesians the cone is $x^2(G_1-G_2)-z^2(G_2-G_3)+\frac{r_2^2}{3G_2}y^2=0$. Hence a plane $Sk\rho=$ const. cuts the cone in an ellipse, or the line parallel to k through the vertex of the cone lies within it.

The cuspidal circle and cuspidal sphero-conic have four points common in the plane infinity.

The form of the focal surface is now evident.

The section by the central plane through i and j has two conjugate points $\pm r_2 j$, and no real cusps. It consists of an oval curve passing through $\pm r'_1 i$ and $\pm r'_2 j$, and no central line in the plane, except j, meets it in more than two real points.

The section by the plane through jk has a pair of real nodes at $\pm r_2 j$, four real cusps on the lines

$$G_2^3 y^2 + G_3^3 z^2 = 0,$$

and meets j and k at $\pm r'_2 j$, and $\pm r'_3 k$.

FIG. 5.

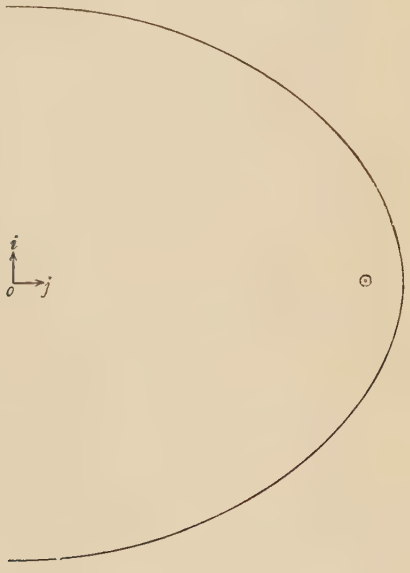

FIG. 4.

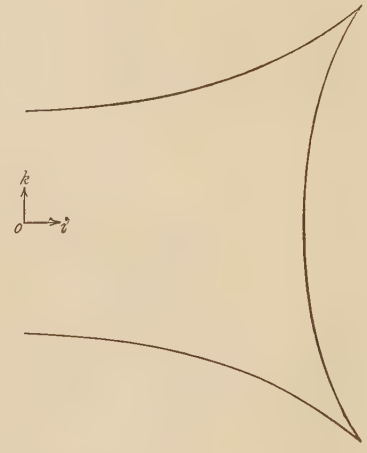

FIG. 6.

Figs. 4, 5, and 6 are halves of the principal central sections of the focal surface when $G_1 = 3$, $G_2 = 1$, $G_3 = -4$.

The section by the plane ki has four real cusps on the lines $G_1^3 x^2 + G_3^3 z^2 = 0$, and no real double point.

These curves have six cusps, four double points, no inflexion, three double tangents, and are of the fourth class.

Conceive the central plane through jk rotated round k. The nodes disappear, and at first the curve has a pair of marked bends nearly meeting

in the neighbourhood of each node in the plane jk, as the section of the tangent cones indicates. These bends straighten out, and the points on the diameter at right angles to k gradually close till they are at a distance r'_1 from the centre.

Secondly, rotating the plane through ij round i, the conjugate points open out into closed curves, and the trace consists of these small ovals surrounded by a larger oval passing through $\pm r_1'i$. On continued rotation the smaller ovals, as well as the larger, expand—the smaller more rapidly than the larger—until the plane touches the cuspidal sphero-conic. Then the smaller and larger ovals touch at points on the sphere $r_0 = \sqrt{-M_2}$, and the smaller ovals simultaneously touch the sphere of radius

$$r' = -\frac{1}{2M_2}\sqrt{-4M_2^3 - 27M_3^2}.$$

The reason of this is, the diameter at right angles to i is about to cut the surface in only two real points, for on further rotation of the plane the points of contact of the ovals (tac-node-cusps) become a pair of cusps, and it has been shown that the six real points in which a central line meets the surface lie outside the small sphere r'. On further rotation of the plane, the distances between the pairs of cusps on the same side of i increase till the plane passes through k, and the two real points on the diameter at right angles to i close in till they coincide with $\pm r'_3 k$.

Thirdly, rotating the plane through jk round j, the cusps retreat from j, uniting on the diameter at right angles to j, which lies on the cone $S\rho\Psi^3\rho = 0$. The section consists of two ovals through both nodes touching an outer oval on the diameter perpendicular to j. As the plane rotates still more, contact ceases. The ovals become more and more rounded in form, and the inner ovals tend to coincide, and do so when the plane coincides with one of the cyclic planes $S\rho(\Psi - G_2)\rho = 0$. The ovals are now circles. After this the inner double circle instantly disappears, leaving nothing but two conjugate points $\pm r_2 j$. The outer oval varies, the semi-diameter at right angles to j shrinking in length from r'_2 to r'_1.

Consideration of the sections of planes through k leads most easily to an appreciation of the form of the included region. If k is supposed to be

vertical, the interior of the surface is like a closed cavern, the roof of which is connected with the floor by two pillars through $\pm jr_2$. The enclosed region is "connected," that is, it is possible to pass from one point in it to any other without cutting through the surface.

If Δu is any small scalar, it may be asked how is the surface $4(M_2-\rho^2)^3 + 27(M_3 - S\rho\Psi\rho)^2 + \Delta u = 0$ situated with respect to the focal surface. The surfaces do not meet in real points since the roots of a real linear vector function cannot be equal and unequal at the same time. It is sufficient, therefore, to consider points on the new surface near $\pm r_2 j$. If the new surface meet the axis j in points $\pm(r_2 + \Delta r_2)j$, on substitution and simple reduction

$$4\left[(r_2+\Delta r_2)^2 - r_2^2\right]^2\left[(r_2+\Delta r_2)^2 - r'^2_2\right] + \Delta u = 0, \text{ or } 36 G_2^2 r_2^2 (\Delta r_2)^2 - 4\Delta u = 0.$$

If Δu is positive, the axis j meets the surface in real points near $\pm r_2 j$. The surface lies wholly within the focal surface, and the roots and axes corresponding to points on it are real. If Δu is negative, the surface lies outside the focal surface.

7. The equation of the tangent plane at ρ may, by Art. 2, be written in the form $S(\varpi - \rho)V\rho_1\rho'_1 = 0$, ρ_1 and ρ'_1 being the axes of $\Phi + V\rho$ and its conjugate, whose roots are equal to g.

The tangent to the sphero-conic g is found by writing $dg = 0$ on passing to a consecutive point, so $S\rho\, d\rho = S\Psi\rho\, d\rho = 0$, and $d\rho$ is parallel to $V\rho^{-1}\Psi\rho$.

These are merely special cases of the tangent plane to a root quadric and the tangent line to a root sphero-conic $\rho^2 = M_3 - g_2g_3 - g_3g_1 - g_1g_2$, $S\rho\Psi\rho = M_3 - g_1g_2g_3$.

The sphero-conics in which a cone $S\rho(\Psi - g_1)\rho = 0$ meets the focal surface, have their double roots given by $2g^3 + M_3 = g_1(3g^2 + M_2)$. If the roots are real, $D = (M_3 - g_1M_2)M(g_1)$—the discriminant of the cubic—is negative. Hence may be deduced the conditions derived by the aid of the semi-cubical parabola. Again, the cubic may be written

$$(2g+g_1)(g-g_1)^2 = -M(g_1) = -S\rho_2(\Psi - g_1)\rho_2 = T^2\rho_2(g_2 - g_1),$$

if $\rho_2 = V\rho_1^{-1}\Psi\rho_1$, and g_2 is its root, while $g_1 = S\rho_1^{-1}\Psi\rho_1$, ρ_1 being parallel to

an edge of the cone. In this notation, $D = T^2\rho_2(g_1-g_2)[T^2\rho_2(g_1-g_2)+g_1^3]$. For real roots

$$\frac{g_1^3}{g_2-g_1} > T^2\rho_2 > 0\,;\ \text{or, since } T^2\rho_2+g_1^2 = \frac{T^2\Psi\rho_1}{T^2\rho_1},\ \frac{g_1^2g_2}{g_2-g_1} > \frac{T^2\Psi\rho_1}{T^2\rho_1}.$$

In the present notation, the tangential equation of the focal surface, as given in Art. 4, is $\lambda^2(M_3S\lambda\Psi^{-1}\lambda-1)-\frac{1}{4}S^2\lambda\Psi\lambda = 0$; or, in another form

$$\lambda^2[M_3S\lambda\Psi^{-1}\lambda-1+gS\lambda\Psi\lambda+g^2\lambda^2]-\tfrac{1}{4}S^2\lambda(\Psi+2g)\lambda = 0.$$

This expresses that the planes of the developable circumscribed to a root quadric, along its curve of contact with the focal surface, are parallel to the planes enveloping a quadric cone $S\rho(\Psi+2g)^{-1}\rho = 0$.

Or, again, the focal surface is

$$[M_3S\lambda\Psi^{-1}\lambda-1+gS\lambda\Psi\lambda+g^2\lambda^2][M_3S\lambda\Psi^{-1}\lambda-1+g'S\lambda\Psi\lambda+g'^2\lambda^2]$$
$$= [M_3S\lambda\Psi^{-1}\lambda-1+\tfrac{1}{2}(g+g')S\lambda\Psi\lambda+gg'\lambda^2]^2;$$

or, the developables circumscribed to two root quadrics along their curves of contact with the focal surface envelope a quadric.

The edges of the quadric cone are parallel to the tangents to the focal surface conjugate to the tangents along its curve of contact with the hyperboloid. In general the tangent plane, at a point $\rho+d\rho$, differs from the tangent plane $S(\varpi-\rho)(\Psi-g)\rho = 0$ by $S(\varpi-\rho)[(\Psi-g)d\rho-dg\,\rho] = 0$. The tangent conjugate to $d\rho$ is consequently parallel to $M(g)(\Psi-g)^{-1}V\rho\,d\rho + dg\,V\rho\Psi\rho$. If $dg = 0$, $d\rho$ is parallel to $V\rho^{-1}\Psi\rho$, and the conjugate direction is parallel to $(\Psi-g)^{-1}[\Psi-S\rho^{-1}\Psi\rho]\rho = (\Psi-g)^{-1}\left[\Psi-\frac{M_3+2g^3}{M_2+3g^2}\right]\rho$, or to $(\Psi-g)(\Psi+2g)\rho$. The equation of the lines of curvature is

$$M(g)S\rho\,Vd\rho(\Psi-g)^{-1}d\rho+\frac{1}{6g}S\rho\,d\rho\,S\rho\Psi\rho\,d\rho = 0.$$

The inverse of the reciprocal surface to a unit sphere is the locus of feet of perpendiculars on tangent planes to the surface. Its equation is

$$M_3\rho^2S\rho\Psi^{-1}\rho-\rho^6-\tfrac{1}{4}S^2\rho\Psi\rho = 0,$$

a sextic surface with a quadruple point at the origin, the tangent cone at

which consists of four imaginary planes, and with a cuspidal circle at infinity.

In Cartesian co-ordinates the recipocal of the focal surface is

$$f = r'_1{}^2 l^4 + r'_2{}^2 m^4 + r'_3{}^2 n^4 - 2r'_2 r'_3 m^2 n^2 - 2r'_3 r'_1 n^2 l^2 - 2r'_1 r'_2 l^2 m^2 - (l^2+m^2+n^2)p^2 = 0,$$

λ being $(il+jm+kn)p^{-1}$, and r'_1 as usual, $\frac{1}{2}(G_2 - G_3)$. Or this is $\Pi(\sqrt{r'_1}\, l \pm \sqrt{r'_2}\, m \pm \sqrt{r'_3} n) = (l^2+m^2+n^2)p^2$, Π denoting the product of the four indicated functions.

From the first form,

$$\tfrac{1}{2} \cdot \frac{df}{dl} = 2r'_1 l(r'_1 l^2 - r'_2 m^2 - r'_3 n^2) - lp^2, \quad \tfrac{1}{2}\frac{df}{dp} = -p(l^2+m^2+n^2),$$

and the co-ordinates of the singular tangent planes are

$$l = m = n = 0; \quad p = l = 0, \ \sqrt{r'_2} m \pm \sqrt{r'_3} n = 0; \quad p = m = 0,$$
$$\sqrt{r'_3} n \pm \sqrt{r'_1}\, l = 0; \quad p = n = 0, \ \sqrt{r'_1}\, l \pm \sqrt{r'_2}\, m = 0;$$

and $p = \pm l r_1 = \pm m r_2 = \pm n r_3$, where $r_1{}^2 = 4r'_2 r'_3$.

The first is the plane at infinity. The second group consists of the six cyclic planes, and the third group consists of the eight planes through a double point on each of the axes.

The equation of the plane conic, in which the plane of the third group having all signs positive touches the surface, is

$$\Sigma r'_1{}^3 l^2 - 2\Sigma (r'_2 r'_3)^{\frac{3}{2}} mn - 2p(r'_1 r'_2 r'_3)^{\frac{1}{2}} \Sigma r'_1{}^{\frac{1}{2}} l = 0,$$

or, in point co-ordinates, the conic lies in the plane $\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3} = 1$, and on the cone $r_1{}^3 yz + r_2{}^3 zx + r_3{}^3 xy = 0$. Two conics lie on each cone. The planes and conics are imaginary.

The second form of the equation illustrates the nature of the four points at infinity common to the cuspidal circle and sphero-conic.

8. In order to give some idea of the shapes of the pedal and nodal surfaces, $S\rho\Psi\rho\Psi^2\rho = \pm V^2\rho\Psi\rho$, I shall notice some of their common properties.

The origin is a triple point on both, and each has the lines i, j, and k

double. Also they have common the imaginary quartic $V^2\rho\Psi\rho = 0$ in the plane infinity. Consequently, they have no other common point.

If $\rho' = i\rho i^{-1}$, the vector ρ' satisfies the equations as well as ρ, or the reflexion of any point on one of the surfaces with respect to the lines i, j, and k, is also on the surface. For, $\rho' = -\rho + 2iSi^{-1}\rho$, so $\Psi\rho' = -\Psi\rho + 2G_1 iSi^{-1}\rho$ $= -\Psi\rho + 2iSi^{-1}\Psi\rho = i\Psi\rho i^{-1}$, and in like manner $\Psi^2\rho' = i\Psi^2\rho i^{-1}$. Also $V\rho'\Psi\rho'$ $= Vi\rho i^{-1}i\Psi\rho i^{-1} = i(V\rho\Psi\rho)i^{-1}$. Hence $S\rho'\Psi\rho'\Psi^2\rho' = S\rho\Psi\rho\Psi^2\rho$, and $V^2\rho'\Psi\rho'$ $= V^2\rho\Psi\rho$. This proposition may be stated without reference to the surfaces in the form $i(V\rho^{-1}\Psi\rho + x\rho)i^{-1} = V\rho'^{-1}\Psi\rho' + x\rho'$, or the reflexion of any line of the congruency to i, j, or k is also a line of the congruency. Corresponding points on these lines have equivalent properties as respects roots, meeting the focal surface, &c. The surfaces belong to a general type called Steiner's Quartics. Any plane meets one of the surfaces in a trinodal quartic, the nodes being on the double lines. Any tangent plane meets it in a pair of conics which intersect on the double lines and at the point of contact. The reciprocal of the surface is a cubic with four double points. Or through any line three tangent planes can be drawn to the surface, and four planes touch the surface along a conic. (See Salmon's "Three Dimensions," Art. 523.)

If $\rho = ix + jy + kz$, the pedal surface may be written,

$$2xyz\,r'_1r'_2r'_3 = r'^2_1y^2z^2 + r'^2_2z^2x^2 + r'^2_3x^2y^2,$$

where $r'_1 = \frac{1}{2}(G_2 - G_3)$; $r'_2 = \frac{1}{2}(G_3 - G_1)$; $r'_3 = \frac{1}{2}(G_1 - G_2)$.

If $y = r\cos\theta$, and $z = r\sin\theta$, this becomes, on discarding the factor r^2,

$$2x\sin\theta\cos\theta\,r'_1r'_2r'_3 = r^2r'^2_1\sin^2\theta\cos^2\theta + x^2(r'^2_2\sin^2\theta + r'^2_3\cos^2\theta),$$

which, if θ is given, is the equation of an ellipse in the plane through i, which makes the angle θ with j. The ellipse meets i at the point $ix_1 = i2\dfrac{\sin\theta\cos\theta r'_1r'_2r'_3}{r'^2_2\sin^2\theta + r'^2_3\cos^2\theta}$, and passes through the origin. Since $r'_1r'_2r'_3$ is negative, x_1 is negative when $\sin 2\theta$ is positive, and positive when $\sin 2\theta$ is negative. The real part of the surface is, consequently, finite, as the conics in planes through the double line are ellipses. On account of the variation of sign of x_1 with that of $\sin 2\theta$, the surface is seen to have a

twisted structure like a two-bladed screw-propeller. It consists of four lobes connected by the double lines. The maximum values of x_1 are $\pm r'_1$. These answer to the extreme points in which the axis i meets the focal surface. Outside these points the double line i exists as a series of conjugate points isolated from real parts of the surface.

The vectorial equation of the surface is

$$\rho = i\frac{r'_1r'_2r'_3\sin\theta\cos\theta}{r'^2_2\sin^2\theta + r'^2_3\cos^2\theta}(1+\cos u) + \frac{r'_2r'_3(j\cos\theta + k\sin\theta)}{\sqrt{(r'^2_2\sin^2\theta + r'^2_3\cos^2\theta}}\sin u,$$

u being the eccentric angle of a point on the ellipse. If $\tan v = \frac{r'_2}{r'_3}\tan\theta$,

$$\rho = ir'_1\sin v\cos v(1+\cos u) + (jr'_2\cos v + k\,r'_3\sin v)\sin u.$$

For any value of u, the curve given by varying values of v lies on the elliptic cylinder $\frac{y^2}{r'^2_2} + \frac{z^2}{r'^2_3} = \sin^2 u$, and the rectangular hyperbolic paraboloid $\frac{yz}{r'_2r'_3} = \frac{x}{r'_1}(1-\cos u)$. The real part of the surface is consequently included by the three elliptic cylinders $\frac{y^2}{r'^2_2} + \frac{z^2}{r'^2_3} = 1$; $\frac{z^2}{r'^2_3} + \frac{x^2}{r'^2_1} = 1$, and $\frac{x^2}{r'^2_1} + \frac{y^2}{r'^2_2} = 1$. It cuts, however, the ellipsoid $\frac{x^2}{r'^2_1} + \frac{y^2}{r'^2_2} + \frac{z^2}{r'^2_3} = 1$. For $\varpi = \pm ir'_1\cos u + (jr'_2\cos v + kr'_3\sin v)\sin u$ is the vector equation of the ellipsoid; and if $\sin v\cos v(1+\cos u) = \pm\cos u$ be used to eliminate u or v from either ρ or ϖ, the vectorial equation of the curve of intersection may be found.

If $\tan^2\theta = \frac{r'_3}{r'_2}$, the conic in the plane through i is a circle. The planes thus determined are the cyclic planes of the quadric $S\rho\Psi\rho = M_3$. In the real cyclic planes the diameter of the circle is

$$2\sqrt{r'_3r'_1} = r_2 = \sqrt{(G_2 - G_3)(G_1 - G_2)}.$$

This circle touches the circle of the focal surface along which it is touched by the cyclic plane.

The lines of the congruency in the cyclic planes are found by drawing any chord of this circle through the origin, and drawing a line at right angles to the chord at its extremity. The planes are singular with respect

to the congruency (see Art. 3), and the points $\pm r_2 j$ are singular also, since an infinite number of lines of the congruency pass through them.

The section made by a plane parallel to one of the principal planes resembles a distorted lemniscate. Its equation is that obtained by regarding x as constant, and r and θ variable in the equation given for the conic in the plane $\theta = \text{const.}$ Two of the double points are at infinity, and the tangents at the double point xi are given by $x = \frac{2yz\, r'_1 r'_2 r'_3}{r'^2_2 z^2 + r'^2_3 y^2}$. If x varies, these tangents generate a surface like a cylindroid.

To determine the singular tangent planes, if

$$r'^2_1 y^2 z^2 + r'^2_2 z^2 x^2 + r'^2_3 x^2 y^2 = 2xyz\,(Ax + By + Cz)$$

is a perfect square, $Ax + By + Cz = r'_1 r'_2 r'_3$ touches the surface along a conic. The planes are, consequently, $\pm \frac{x}{r'_1} \pm \frac{y}{r'_3} \pm \frac{z}{r'_3} = 1$, all or only one sign being negative. They lie on the cones $\pm r'_1 yz \pm r'_2 zx \pm r'_3 xy = 0$.

If the plane is merely an ordinary tangent plane, the determinant

$$\begin{vmatrix} -r'^2_1 & C & B \\ C & -r'^2_2 & A \\ B & A & -r'^2_3 \end{vmatrix} = 0,$$

as the quartic breaks up into a pair of quadric cones. If one of these cones is $\frac{r'_1}{p_1} yz + \frac{r'_2}{p_2} zx + \frac{r'_3}{p_3} xy = 0$, the other is $r'_1 p_1 yz + r'_2 p_2 zx + r'_3 p_3 xy = 0$, and $-2A = \left(\frac{p_3}{p_2} + \frac{p_2}{p_3}\right) r'_2 r'_3$, &c.

If one cone is $S\alpha V\rho\Psi\rho = 0$, the other is $S\alpha_{-1} V\rho\Psi\rho = 0$, if when $\alpha = ip_1 + jp_2 + kp_3$, $\alpha_{-1} = \frac{i}{p_1} + \frac{j}{p_2} + \frac{k}{p_3}$. α_{-1} is parallel to $(\Psi^2 + \frac{1}{3} M_2) V\alpha\Psi\alpha$, as may be easily verified.

9. The lines of the congruencies may be drawn by the aid of a single root quadric, $S\rho\Psi\rho = M_3$ for instance. If R is the length of any central radius, $M_3 = -R^2 S\rho^{-1}\Psi\rho = -R^2 g$, where g is the root of the line parallel

to ρ. If P is the perpendicular on the tangent plane to the quadric at $RU\rho$,

$$\Psi U\rho = \frac{M_3}{PR} U\Psi\rho \text{ and } V\rho^{-1}\Psi\rho = -VU\rho\Psi U\rho = \frac{M_3}{PR} VU\rho \frac{1}{U\Psi\rho}$$

$$= \frac{M_3}{PR} \sin \widehat{RP}\, UVU\rho \frac{1}{U\Psi\rho}$$

since $\frac{1}{U\Psi\rho}$ is a unit vector along the perpendicular on the tangent plane. This determines the length of the perpendicular on the line of the congruency. Its direction is the second principal axis of the section of the quadric of which ρ is a principal axis. Thus, in any central plane, the two lines of the congruency may be found definitely, except when the plane is cyclic, the directions of the principal axes being in this case indeterminate. In any central plane the lines of the two congruencies form a rectangle. If $\alpha' = V\alpha^{-1}\Psi\alpha$, $\alpha = V\alpha'^{-1}\Psi\alpha'$, the points α, α' and $-(\alpha+\alpha')$ lie on the pedal surface, $-\alpha$, $-\alpha'$ and $+(\alpha+\alpha')$ on the nodal surface, and the points $\pm(\alpha\pm\alpha')$ on a root sphero-conic. If abc are the roots at α', and $a'b'c'$ those at α, two lines through $-\alpha'$ are coplanar with α', and at right angles to α and to one another (Article 2). If the lines are parallel to β and γ, and their respective roots are b and c, $\beta = \frac{(\Psi-c)\alpha'}{c-b}$, $\gamma = \frac{(\Psi-b)\alpha'}{b-c}$. For $(\Psi - V\alpha' - b)(\Psi - V\alpha' - c)\alpha' = 0$, as α' is coplanar with β and γ, and from the rectangle in the plane of β and γ, $\beta+\gamma+\alpha' = 0$. This condition of coplanarity may be written, $(\Psi-b)(\Psi-c)\alpha' = V\alpha'\Psi\alpha' = \alpha'^2\alpha$, so $\beta = \frac{\alpha'^2(\Psi-b)^{-1}\alpha}{c-b}$, and $\gamma = \frac{\alpha'^2(\Psi-c)^{-1}\alpha}{b-c}$. The line through $(\alpha+\alpha')$ parallel to $\alpha\alpha'$ is given with proper tensor by

$$\frac{V(\alpha+\alpha')\Psi(\alpha+\alpha')}{(\alpha^2+\alpha')^2} = \frac{V(\alpha+\alpha')(\alpha\alpha+\alpha'\alpha')}{\alpha^2+\alpha'^2} = \frac{(\alpha'-\alpha)\alpha\alpha'}{\alpha^2+\alpha'^2},$$

as $\Psi\alpha + \alpha'\alpha = a\alpha$ and $\Psi\alpha' + \alpha\alpha' = \alpha'\alpha'$. The direction of the third line through $\alpha + \alpha'$ is found by operating by

$$[\Psi + V(\alpha+\alpha') - \alpha][\Psi + V(\alpha+\alpha') - \alpha']$$

on an arbitrary vector, $\alpha\alpha'$, suppose. Now

$$[\Psi + V(\alpha+\alpha')]\alpha\alpha' = -(\alpha+\alpha')\alpha\alpha' + 2(\alpha+\alpha')V\alpha\alpha'$$
$$= -(\alpha+\alpha')\alpha\alpha' + 2(\alpha'\alpha^2 - \alpha\alpha'^2).$$

So $(a+2a')(a'+2a)\,\alpha\alpha' + 2\alpha'^2(a+2a')\alpha - 2\alpha^2(a'+2a)\alpha'$ is the direction of the line. Operating by $V\alpha$, the direction of the line through $-(a+a')$, whose root is a', is found to be $(\Psi+\frac{1}{2}a)\,\alpha'$, and the line whose root is a is parallel to $(\Psi+\frac{1}{2}a')\,\alpha$. The pedal points on these lines are

$$\frac{V(\Psi+\frac{1}{2}a)\alpha'\Psi(\Psi+\frac{1}{2}a)\alpha'}{\left[(\Psi+\frac{1}{2}a)\alpha'\right]^2} = M(-\tfrac{1}{2}a)\,\frac{(\Psi+\frac{1}{2}a)^{-1}\alpha}{\alpha'^{-2}\left[(\Psi+\frac{1}{2}a)\alpha'\right]^2}$$

$$= M(-\tfrac{1}{2}a)\,\frac{(\Psi+\frac{1}{2}a)^{-1}\alpha}{(a'+\frac{1}{2}a)^2-\alpha^2},$$

with a similar expression in a' and α' for the other line. Again,

$$\frac{V(\Psi+\frac{1}{2}a)^{-1}\alpha\,\Psi(\Psi+\frac{1}{2}a)^{-1}\alpha}{\left[(\Psi+\frac{1}{2}a)^{-1}\alpha\right]^2}$$

determines a point on the pedal surface.

Certain invariantal relations connecting abc, $a'b'c'$ may be derived; since

$$\alpha'^2 = M_2-(bc+ca+ab), \quad \alpha^2 = M_2-(b'c'+c'a'+a'b'),$$

and $$\alpha^2+\alpha'^2 = M_2+a^2+aa'+a'^2 = M_2-(a'a''+a''\,a+aa'),$$

where $$a+a'+a''=0, \text{ therefore } M_2 = bc+b'c'+aa'.$$

Again, $a\alpha^2+a'\alpha'^2 = M_3 - aa'a''$ gives $M_3 = abc+a'b'c'-aa'a''$, using the easily derived relations $\alpha^2=(a'-b)(a'-c)$ and $\alpha'^2=(a-b')(a-c')$.

Referring to the end of Art. 4, comparison of the magnitudes of the quantities t_1^2 and t'^2_1 with unity shows that both cannot be greater than unity, though both may be less. Both pairs of focal points cannot lie outside the rectangle here considered.

10. Instead of taking the plane through the origin, let it be drawn through a fixed point ϵ. In the plane are two lines of each congruency, and since a line of one congruency cuts the lines of the other it meets at right angles, which do not lie on the same root quadric; if the lines of the first congruency are parallel to ρ_1 and ρ_2, with roots g_1 and g_2, the lines of the second congruency are ρ'_1 and ρ'_2 of roots g_1 and g_2, and ρ'_1 is perpendicular to ρ_2, and ρ'_2 to ρ_1. Taking the origin at ϵ, if ρ is the vector to the point of intersection of ρ_1 and ρ_2, $\rho = x\rho_1+y\rho_2$, so $(\psi+V\rho-g_1)(\psi+V\rho-g_2)\rho=0$,

where $\psi = \Psi + V\epsilon$. Hence, $S\rho\psi\rho\psi^2\rho + V^2\rho\psi\rho = 0$ is the locus of intersection of ρ_1 and ρ_2, and in like manner, $S\rho\psi'\rho\psi'^2\rho - V^2\rho\psi'\rho = 0$ is the locus of intersection of ρ'_1 and ρ'_2. These quartic surfaces are analogous to the pedal and nodal surfaces, the double lines of the first are the axes of ψ, and those of the second the axes of $\psi' = \Psi - V\epsilon$. The locus of intersection of ρ_1 and ρ'_1 may be found thus: if ρ is the vector from ϵ to the point of intersection, $M(g_1) = S(\epsilon+\rho)(\Psi-g_1)(\epsilon+\rho)$. But $\rho = x\rho_1 + y\rho'_1$, so $S(\epsilon+\rho)(\Psi-g_1)\rho = 0$. The locus is got by eliminating g_1.

The lines of one congruency which meet an arbitrary line generate a quintic surface on which the line is triple, since through any point three lines of the congruency pass. The surface is a quintic, since any plane through the line contains two lines of the congruency which, with the line counted three times, constitute a curve of the fifth degree. If $\epsilon + t\alpha$ is the line, and if two lines of the congruency coplanar with it are ρ_1 and ρ_2, $(\psi - t)\rho = (g_1 - t)x_1\rho_1 + (g_2 - t)x_2\rho_2 = y\alpha$ for a proper value of t;

$$\rho\,(= x_1\rho_1 + x_2\rho_2 = t_1(\alpha + y_1\rho_1) = t_2(\alpha + y_2\rho_2)),$$

being the vector from ϵ to the point of intersection of ρ_1 and ρ_2. y is determined, since ρ satisfies $S\rho\psi\rho\psi^2\rho + V^2\rho\psi\rho = 0$, by the equation

$$S(\psi-t)^{-1}\alpha\,\alpha\psi\alpha + y\,V^2(\psi-t)^{-1}\alpha\alpha = 0, \text{ or } m(t)S\alpha\psi\alpha\psi^2\alpha + y\left[(\psi-t)V\alpha\psi\alpha\right]^2 = 0,$$

$m(t)$ being the third invariant of $(\psi - t)$. To avoid needless symbols, assume the tensor of α to satisfy $S\alpha\psi\alpha\psi^2\alpha = V^2\alpha\psi\alpha$, and let $\alpha' = V\alpha^{-1}\psi\alpha$. The locus of ρ, the point of intersection of generators of the quintic coplanar with the triple line, is therefore,

$$\rho = -\,m(t)\,\frac{(\psi-t)^{-1}\alpha}{\alpha'^{-2}\left[(\psi'-t)\,\alpha'\right]^2} = -\,\frac{\psi^2\alpha + t\psi\alpha + (m_2 + t^2)\,\alpha}{\alpha'^{-2}\left[\psi'-t)\alpha'\right]^2},$$

an *ellipse* meeting the line at $\epsilon - \alpha$. This ellipse lies on the cone $S\alpha\rho\psi\rho = 0$, and lies in a tangent plane to the nodal quartic (see Art. 8). The quintic has, consequently, as singularities, a nodal ellipse and a triple line.

The equation of the nodal ellipse may be otherwise expressed, since, as is easily verified, $V\alpha'^{-1}\psi'\alpha' = \alpha$. Identically, $\psi\alpha + \alpha'\alpha = a\alpha$, $\psi'\alpha' + \alpha\alpha' = a'\alpha'$, a and a' being roots of the lines of the congruency parallel to α and α',

so $\alpha'^{-2}[(\psi'-t)\alpha']^2 = (a'-t)^2 - a^2$; and if $\frac{t-a'}{T\alpha} = \tan u$, the equation of the ellipse is,

$$\rho = -\frac{1}{2T^2\alpha}[\psi^2\alpha + a'\psi\alpha + (m_2 + a'^2 + T^2\alpha)\alpha]$$
$$-\frac{1}{2T\alpha}(\psi + 2a')\alpha \sin 2u + \frac{1}{2T^2\alpha}[\psi^2\alpha + a'\psi\alpha + (m_2 + a'^2 - T^2\alpha)\alpha]\cos 2u,$$

referred to a pair of conjugate semi-diameters.

If $\epsilon = 0$, $\Psi^2\alpha = \Psi(a\alpha' + a\alpha) = a^2\alpha + a\,a\alpha' + a'\Psi\alpha + \Psi a'\alpha$
$= \alpha(a^2 - a'^2) + a'a^2 - a'a\alpha'$;

and $\Psi^2\alpha + a'\Psi\alpha + (M_2 + a'^2)\alpha = \alpha' a^2 + a^3$. So the conic is

$$\rho = \frac{1}{2}a' - \frac{1}{2T\alpha}[(a + 2a')\alpha + a\alpha']\sin 2u - \frac{1}{2}(a' + 2a)\cos 2u.$$

Similar reduction may be made if the origin ϵ is specially assigned on an arbitrary line.

If α is an axis of ψ, $(\psi - t)\rho = y\alpha$ shows that the locus of ρ is along the line α. The method here employed does not admit of the determination of the length of the locus on the line. It is, however, evidently the part of the double line which "really" lies on the nodal surface of ϵ.

11. The classification of these quintic surfaces may be based on—1st, the number of real points in which the triple line meets the focal surface; 2nd, the number of real tangent planes which can be drawn through the line to the focal surface. According to the first method there are three, two, one, or no finite portions of the line through which three real sheets of the surface pass. The second method deals with the nodal ellipse, through every point of which two real sheets may pass, or which may be divided into two or four portions, through one or two of which no real sheet passes. As a plane through the triple line revolves, the real lines in it become coincident when it touches the focal surface, and after contact become imaginary.

The director cone of generators of a quintic is a cubic. If a line of the congruency, parallel to ρ, pass through $\epsilon + t\alpha$, $\psi\rho + tV\alpha\rho = g\rho$, and $SV\alpha\rho V\rho\psi\rho = 0$, is the director cone. This may be written

$$S\rho\Psi\rho S\alpha\rho - \rho^2 S\rho\psi'\alpha = 0,$$

and the cone is generated by the corresponding intersections of the homographic systems $S\rho\Psi\rho - g\rho^2 = 0$, $S\rho\psi'\alpha - gS\alpha\rho = 0$. If the plane touch the quadric cone, $S(\psi' - g)\alpha(\Psi - g)^{-1}(\psi' - g)\alpha = 0$; or,

$$S(\psi' - g)\alpha[\alpha - (\Psi - g)^{-1}V\epsilon\alpha] = 0;\text{ or, } S\alpha(\Psi - g)\alpha + SV\epsilon\alpha(\Psi - g)^{-1}V\epsilon\alpha = 0;$$

$$\text{or again, } [M(g) - S\epsilon(\Psi - g)\epsilon]S\alpha(\Psi - g)\alpha + S^2\epsilon(\Psi - g)\alpha = 0.$$

This quartic in g determines the tangent planes through the line $V\alpha\psi'\alpha$ on the cubic to the cubic, and also the root quadrics touched by the line $\epsilon + t\alpha$. If the roots are all real, or all imaginary, the cubic is bipartite, and if two are real, the cubic is unipartite.* The quintic is in like manner bipartite or unipartite. For, if the cone is unipartite, there is continuity in change of direction from one generator to any other.

If ϵ is parallel to $V\alpha^{-1}\Psi\alpha$, the roots are all real if ϵ lies within the focal surface; and if ϵ is outside the focal surface, only two roots are real. The tangent planes through $V\alpha\psi'\alpha$ are, in this case, tangent along α and the three axes of ψ.

If no real tangent plane can be drawn to the focal surface through $\epsilon + t\alpha$, the line α lies on the oval of the cone. If four real tangent planes exist, the cone is again bipartite and α lies on the infinite branch; and if two of the tangent planes are imaginary, the cone is unipartite.

Having regard to the quintic surface, the quartic in g determines the roots of consecutive generators having the same root. A tangent plane to the focal surface, through the triple line, determines, on the other hand, consecutive generators coplanar with the triple line. Suppose ϵ parallel to $V\alpha^{-1}\Psi\alpha$, and its extremity within the focal surface. If $a = S\alpha^{-1}\Psi\alpha$, the quartic in g has real roots g_1, g_2, g_3, and a. The cone is bipartite; therefore either four or no real tangent planes can be drawn through α.

* See Salmon's *Higher Plane Curves*, Art. 167.

The lines of contact of tangent planes through $V\alpha^{-1}\psi'\alpha$ must be two on the oval, and two on the infinite branch. If no real tangent plane can be drawn through α, α lies on the oval. Now, the roots of lines through $\epsilon + t\alpha$ satisfy $S(\epsilon + t\alpha)(\Psi - g)(\epsilon + t\alpha) = M(g)$, or,

$$t^2(g-a)T^2\alpha = (g_1-g)(g_2-g)(g_3-g); \text{ and if } a > g_1 > g_2 > g_3,$$

g must satisfy one of the inequalities $a > g > g_1$, or $g_2 > g > g_3$, since t^2 cannot be negative for real lines. Thus, if α lies on the oval, the roots of lines on the oval satisfy the inequality $a > g > g_1$ and the roots of lines on the infinite branch $g_2 > g > g_3$. Also, the root of $V\alpha^{-1}\psi'\alpha$ must satisfy this latter inequality. The different possible relative positions of α and $V\alpha^{-1}\psi'\alpha$ on the cone, may thus be examined in detail. If ϵ is zero, the root a must satisfy the inequality $G_1 > a > G_3$. Of course, when ϵ is parallel to $V\alpha^{-1}\psi\alpha$, the line $\epsilon + t\alpha$ touches a root sphero-conic (see Art. 7). In general, the points of contact of $\epsilon + t\alpha$ with root quadrics are given by elimination of g between

$$M(g) = S(\epsilon + t\alpha)(\Psi - g)\epsilon \quad \text{and} \quad S(\epsilon + t\alpha)(\Psi - g)\alpha = 0.$$

12. Before considering analytically the system of director cones of quintic surfaces whose triple lines pass through the fixed point ϵ, it is well to investigate, geometrically, some of the peculiarities of the quintic surfaces.

If the triple line do not meet the focal surface in real points, one, and only one, real generator passes through a point on it. It is, therefore, possible by continuous variation to pass from any one generator of the quintic to any other. The quintic and director cone are therefore unipartite, and two real tangent planes can be drawn through the triple line to touch the focal surface. It may be remarked, that these planes are torsally tangent to the quintic, as they contain two consecutive generators.

If the triple line move so as to touch the focal surface, through the point of contact springs up an isolated double line on the quintic. This consists of a pair of coincident lines of the congruency.

If the line move on so as to cut the surface in two points, the quintic

is bipartite. The part of the quintic that has arisen from the isolated double line is isolated from the rest; a generator of this part cannot, by continuous variation, be made to coincide with a generator of the other part. It is evidently possible to draw a pair of torsal tangent planes through the triple line to this part of the surface. Four real tangent planes can be drawn to the focal surface through the triple line, since when the triple line touched the focal surface, the plane through it and the isolated double generator did not arise from coincidence of the existing pair of tangent planes.

On the other hand, if the line have moved through the cuspidal sphero-conic, so as to cut the surface in two real points, since the three lines of the congruency through a point on the cuspidal sphero-conic are coincident (their root being zero), the double line is not isolated from the rest of the surface, and the surface still remains unipartite. Tracing the variation of a generator through a point on the triple line, outside the focal surface, its point of intersection with the triple line passes through the first point of intersection of the triple line with the focal surface on to the second point, turns back to the first point, and thence back again through the second point, and off to infinity.

Considering these two classes of quintics separately, the triple line of the bipartite quintic may move so as to touch the surface. No isolated line can come into existence, as the quintic cannot be tripartite since the director cone cannot. The double line must arise from union of the two sheets; or, in other words, the line must now touch the focal surface internally—that is, between its two points of intersection with it. After contact, the quintic becomes unipartite. If A', B' are the new points of intersection of the triple line with the focal surface, and A and B the old points, a generator meeting the triple line, outside the focal surface, moves through A to A', back to A, forward to B, back to B', and forward to infinity.

The unipartite quintic may develope an isolated double line, and become bipartite; or its sheet may cut itself, and thus become bipartite; or it may remain unipartite, if the triple line intersect the cuspidal sphero-conic a second time.

The system of plane cubics $S(\epsilon + t\alpha)(\Psi - u)(\epsilon + t\alpha) = M(u)$, where t and u are Cartesian co-ordinates, may be taken as the basis of classification of various forms of quintic surfaces.

13. If ϵ is given while α varies, the system of cones

$$S\rho\,\Psi\rho\,S\alpha\rho - \rho^2 S\rho\,\psi'\alpha = 0,$$

has seven common lines. Four of these are the lines of intersection of $S\rho\,\Psi\rho = \rho^2 = 0$; and three are the axes of $\Psi + V\epsilon$. The line α is the coresidual of these seven lines, since if ρ is put equal to $u\alpha + v\beta$, the equation of the cone gives a symmetrical expression in α and β

$$SV\alpha\beta V(u\alpha + v\beta)\psi(u\alpha + v\beta) = 0.$$

So the lines in the plane of α and β are the $8th$ and $9th$ lines of intersection of the cones of α and β. In Salmon's *Higher Plane Curves*, Art. 263, it is proved that, for a system of cubics through seven fixed points, the locus of the coresiduals of these points, on the nodal cubics of the system, is a curve of the fourth class and twelfth degree, with 28 double points, and 24 cusps.

The quartic in g, determining the tangent planes through $V\alpha\psi'\alpha$, has a double root when the cone has a double line, and by forming its discriminant the cone locus of coresiduals may be found. It is simpler, perhaps, to consider the reciprocal cone which is of the fourth degree. In the Article cited, it is proved that the locus of coresiduals is the envelope of the common tangent plane to two cubics of the system which touch. So, repeating the analysis of Art. 267, of the *Higher Plane Curves*, and expressing that the $8th$ and $9th$ lines common to the cones of α and β coincide, $4SV\alpha\beta V\alpha\psi\alpha\, SV\alpha\beta V\beta\psi\beta - S^2V\alpha\beta V(\alpha\psi\beta + \beta\psi\alpha) = 0$. If $\alpha = l\rho_1 + m\rho_2 + n\rho_3$, and $\beta = l'\rho_1 + m'\rho_2 + n'\rho_3$, while $\lambda = V\alpha\beta = x\rho_1 + y\rho_2 + z\rho_3$, ρ_1, ρ_2, ρ_3 being the axes of ψ, the equation becomes,

$$\Sigma(g_2 - g_3)^2 x^2 (mn' - m'n)^2 - 2\Sigma(g_3 - g_1)(g_1 - g_2)yz(nl' - n'l)(lm' - l'm) = 0.$$

Now, $V\alpha\beta = \Sigma V\rho_2\rho_3(mn' - m'n)$; and if, for the sake of conciseness, the tensors of ρ_1, ρ_2, and ρ_3, and those of ρ'_1, ρ'_2, ρ'_3 (the axes of ψ'), are chosen so that $\rho_1 = V\rho'_2\rho'_3$, and, reciprocally, $\rho'_1 = V\rho_2\rho_3$, viz., by assuming

$S\rho_1\rho_2\rho_3 = -1 = S\rho'_1\rho'_2\rho'_3$, it follows at once that $mn' - m'n = -S\rho_1\lambda$, and $x = -S\rho'_1\lambda$. The reciprocal of the cone of coresiduals takes the symmetrical form,

$$\Sigma(g_2-g_3)^2 S^2\rho_1\lambda S^2\rho'_1\lambda - 2\Sigma(g_3-g_1)(g_1-g_2)S\rho_2\lambda S\rho'_2\lambda S\rho_3\lambda S\rho'_3\lambda = 0.$$

The planes $S\rho_1\lambda = 0$, $S\rho'_1\lambda = 0$, &c., are bitangent planes to this cone. Reciprocally, ρ_1, ρ'_1, &c., are double edges on the reciprocal cone—the duodecimic. The roots of ψ' being the same as those of ψ, this is unchanged when ϵ is changed into $-\epsilon$. The two systems of cones $SV\alpha\rho V\rho\psi\rho = 0$, and $SV\alpha\rho V\rho\psi'\rho = 0$, have consequently the same locus of coresiduals on nodal cones.

When $\epsilon = 0$, the quartic in λ becomes four planes

$$\sqrt{G_2-G_3}\,Si\lambda \pm \sqrt{G_3-G_1}\,Sj\lambda \pm \sqrt{G_1-G_2}\,Sk\lambda = 0, \quad \text{(See Art. 7.)}$$

normal to the lines $\rho^2 = S\rho\Psi\rho = 0$.

By the method given in Art. 265 of the *Higher Plane Curves*, the whole system of double edges of the duodecimic may be found. This consists in finding the coresiduals of the seven lines on the 21 cubics which consists of a quadric cone, through five of the lines, and a plane through the other two. For instance, it may be thus proved, that ρ'_1 is the coresidual on the cubic $S\rho(\psi - g_1)\rho\, S\rho\rho_2\rho_3 = 0$. This, however, is otherwise obvious.

By the geometrical reasoning in the last Article, it is evident that this duodecimic cone is the tangent cone from the point ϵ in the focal surface.

If the triple line of a quintic is a double edge of the cone, the quintic breaks up into factors unless the double generators through the two points of contact with the focal surface coincide with the triple line. This is the case when the triple line is one of the lines of the congruency through ϵ, and here the line becomes quadruple.

If the triple line is an inflexional tangent to the focal surface, the two sheets of the surface through the double generator have a common tangent plane along that generator. The triple line is a cuspidal edge of the tangent cone.

Through a line of the congruency four real tangent planes can be drawn to the focal surface if its root a satisfy $G_1 > -2a > G_3$. In any case every plane through the line contains two real lines of the congruency as imaginaries enter in pairs. The four tangent planes consist of two pairs of coincident planes.

14. The Jacobian or locus of double edges of the system of director cones, $S\rho\Psi\rho S\alpha\rho - \rho^2 S\rho\psi'\alpha = 0$, may be found thus:—If ρ is a double edge $2\Psi\rho S\alpha\rho - 2\rho S\rho\psi'\alpha = \psi'\alpha\rho^2 - \alpha S\rho\Psi\rho$ as the equation of the tangent plane vanishes. But $(\psi + tV\alpha - g)\rho = 0$, so $2(\Psi - g)\rho S\alpha\rho = (\psi' - g)\alpha\rho^2$, or $\alpha\rho^2 = 2(\psi' - g)^{-1}(\Psi - g)\rho S\alpha\rho$; operating by $S\rho$, $\rho^2 = 2S\rho(\psi' - g)^{-1}(\Psi - g)\rho = 2S\rho(\psi' - g)^{-1}(\psi' - g + V\epsilon)\rho$, or $\rho^2 + 2S\rho(\psi' - g)^{-1}V\epsilon\rho = 0$. This becomes, on expanding and replacing g by $S\rho^{-1}\Psi\rho$; $S\rho(\psi - g_1)\rho\, S\rho(\psi - g_2)\rho\, S\rho(\psi - g_3)\rho = 2\rho^4 S\epsilon\rho\psi^2\rho + 2\rho^2 S\rho\psi\rho\, S\epsilon\rho\psi\rho$. The existence of the double lines, $S\rho\psi\rho = \rho^2 = 0$, and $\rho_1\rho_2\rho_3$ is easily perceived. If $\epsilon = 0$, this reduces to the product of the cyclic planes.

It is, perhaps, simpler to throw the vector equation into either of the forms,

$$(\Psi - g)(\rho\alpha\rho^{-1}) = -V\epsilon\alpha \text{ or } (\psi - g)(\rho\alpha\rho^{-1}) = -2V\epsilon\rho\, S\alpha\rho^{-1}.$$

If ϵ is zero $(\Psi - g)(\rho\alpha\rho^{-1}) = 0$; so $g = G_1$, say, and $\rho\alpha\rho^{-1}$ is parallel to i, ρ lies in a cyclic plane, and bisects the angle between α and i, α lies in the same cyclic plane. Consequently, when $\epsilon = 0$, director cones never have a single double line, but break up into a cyclic plane and a quadric cone. If α is parallel to i, the cone is $Si\rho S\rho(\Psi - G_1)\rho = 0$. Here α lies in two cyclic planes. If α is one of the lines, $\rho^2 = S\rho\Psi\rho = 0$, the cone becomes the three cyclic planes through that line. If ϵ is not zero, but lies in a cyclic plane, an edge of the cone ρ_1 lies in the same plane, and two of the four imaginary lines. The cones consist therefore of this plane and a quadric cone.

All this follows easily from the geometry of Art. 12. For instance, if a line through the centre of the focal surface touch it, it must touch it in two points. Hence, when ϵ is zero, director cones cannot have only one double edge. If ϵ lie on the focal surface the tangent cone breaks up.

15. The two cones $SV\alpha\rho V\rho\psi\rho = 0$, and $SV\alpha\rho V\rho\psi'\rho = 0$, may be considered together. Any plane through their common line α contains, in addition, a pair of generators of each cone, one of each pair being at right angles to one of the other pair, these lines being parallel to coplanar lines of the two congruencies. So if β is any edge of one cone, $\beta' = \beta V\alpha\beta$ is an edge of the other. This property may be expressed in the equivalent form, if β is an edge of the first, $\beta' = V\beta\psi\beta$ is an edge of the second cone, and if β' is an edge of the second, $V\beta\psi'\beta$ is an edge of the first. As an example, if α is an axis of ψ, the second cone is $S\alpha\rho S\rho(\Psi - a)\rho = 0$, and the result of writing in this $\rho V\alpha\rho$ for ρ is $V^2\alpha\rho S\rho V\alpha\rho(\psi - a)(\rho V\alpha\rho) = V^2\alpha\rho S\alpha\rho SV\alpha\rho V\rho\psi\rho$.

If $\rho_1\rho_2\rho_3$ are parallel to axes of $\psi + tV\alpha$ and $\rho'_1\rho'_2\rho'_3$ to axes of the conjugate $\psi' - tV\alpha$, $V\rho'_1V\alpha\rho'_1 = V\rho_2\rho_3V\alpha\rho'_1 = \rho_2S\alpha\rho'_1\rho_3 - \rho_3S\alpha\rho'_1\rho_2$ is the edge of the first cone coplanar with α and ρ'_1. This edge is also coplanar with ρ_2 and ρ_3, and may be denoted by ρ_{23}, and called the complementary of ρ_2 and ρ_3. Thus the complementary of two lines of a triad is coplanar with α and the third line of the conjugate triad. The third line of the first cone coplanar with α and ρ'_1 is easily seen to be $V.\ V\alpha\rho'_1(\psi' - g_1)\alpha$, and the third line on the second cone is $V\alpha\rho'_1V.\ V\alpha\rho'_1(\psi' - g_1)\alpha$, g_1 being the root of ρ'_1.

If any edge ρ'_1 of the second cone is chosen, the plane at right angles to this cuts the first cone in ρ_2, ρ_3, and ρ_{23}. ρ_{23} is distinguished from the other lines as it is coplanar with α and ρ'_1. The plane at right angles to ρ_2 cuts the second cone in ρ'_3, ρ'_1, and ρ'_{31}, and that at right angles to ρ_3 cuts it in ρ'_1, ρ'_2, and ρ'_{12}. Finally, planes at right angles to ρ'_2 and ρ'_3 determine ρ_{31} and ρ_{12} on the first cone. The plane containing ρ_1 and $V.\ V\alpha\rho'_1(\psi' - g_1)\alpha$, the line on the first cone coplanar with ρ'_1 and α, is at right angles to $V\rho_1V.\ V\alpha\rho'_1(\psi' - g_1)\alpha$, or simply to $(\psi' - g_1)\alpha$. This plane passes through the fixed line $V\alpha\psi'\alpha$ on the first cone. Applying a notation to Sylvester's principle of residuals and co-residuals (Weierstrass's notation of Elliptic Functions), an edge of a cubic cone may be defined by a single parameter u, and if three edges are coplanar, $u + u' + u'' = 0$. If the parameters of ρ_2, ρ_3, and ρ_{23}, &c., are u_2, u_3, and u_{23}, &c., and if that of α is u_0, and that of $V\alpha\psi'\alpha$, v_0, $u_2 + u_3 + u_{23} = 0$; and if v_1 is the parameter of the third line of the cone coplanar with ρ_{23} and α, $v_1 + u_{23} + u_0 = 0$. Using

similar equations, $u_1+v_1=u_2+v_2=u_3+v_3=-u_0+u_1+u_2+u_3=-v_0$. Hence $u_1+u_2+u_3=u_0-v_0$; $v_1+v_2+v_3=-2v_0-u_0$; $u_{23}+u_{31}+u_{12}=2v_0-2u_0$; these formulæ conveniently collect the properties lately proved. Again, the planes through corresponding conjugate lines, one on each cone, ρ_1, ρ'_1; ρ_2, ρ'_2; and ρ_3, ρ'_3; intersect in the line $V(\epsilon+t\alpha)\Psi(\epsilon+t\alpha)$. As t varies, this traces out a quadric cone. If ϵ is zero, this cone coincides with the line $V\alpha\Psi\alpha$ and with $V\alpha\psi'\alpha$. The plane through ρ_1 and ρ'_1 contains two edges of this quadric cone. The normal to the plane is $(\Psi-g_1)(\epsilon+t\alpha)$, but if t' is the parameter defining the second edge of the cone which lies in the plane, it is evidently possible to choose g', so that $(\Psi-g'_1)(\epsilon+t'\alpha)$ may coincide with this normal. Consequently, the plane of ρ_1 and ρ'_1 contains a second pair of corresponding conjugate lines ϖ_1 and ϖ'_1.

The six edges common to the quadric cone and the cubic $SV\alpha\rho V\rho\psi\rho=0$ consist of parallels to the four lines of the congruency $\epsilon+t\alpha+yV(\epsilon+t\alpha)^{-1}\Psi(\epsilon+t\alpha)$, through the points in which $\epsilon+t\alpha$ meets the pedal surface, and of the parallels to the two lines of the congruency coplanar with the origin and $\epsilon+t\alpha$. For if $\epsilon+t\alpha$ and $\epsilon+t'\alpha$ are parallel to these, $V(\epsilon+t'\alpha)^{-1}\Psi(\epsilon+t'\alpha)+yV(\epsilon+t\alpha)^{-1}\Psi(\epsilon+t\alpha)$ is one of the lines.

The nine lines common to the two cones, $S\alpha\rho S\rho\Psi\rho=\rho^2(S\rho\Psi\alpha\pm S\rho\alpha\epsilon)$ are $\rho^2=S\rho\Psi\rho=0$, $\rho^2=S\alpha\rho=0$, and the three lines in $S\rho\alpha\epsilon=0$. The second pair of equations determine the lines to the "circular" points in the plane $S\alpha\rho=0$. The third equation gives the line α and the pair of generators in the central plane through the line $\epsilon+t\alpha$.

The cones may also be conceived as generated by the corresponding intersections of $\rho^2=2xS\alpha\rho$ with the pair of quadrics

$$S\rho\Psi\rho=2xS\rho\Psi(\alpha\pm\Psi^{-1}V\alpha\epsilon),$$

that is by the intersections of a sphere with a pair of quadrics of given shape and orientation, expanding in such a manner that they pass through the origin, and that when the centre of the sphere is at $x\alpha$, the centres of the quadrics are at $x(\alpha\pm\Psi^{-1}V\alpha\epsilon)$.

16. The equation of the quintic surface generated by lines of the congruency meeting $\epsilon+t\alpha$, is found by elimination of t, y, and ρ_1 from the equation of a generator $\rho=\epsilon+t\alpha+y\rho_1$, by the aid of the relation

$(\psi + tV\alpha - g_1)\rho_1 = 0$. Hence $(\psi + tV\alpha - g_1)(\rho - \epsilon - t\alpha) = 0$, and from this the scalars t and g_1 may be eliminated in two principal ways. If the origin is taken at ϵ, $V(\rho - t\alpha)[\psi(\rho - t\alpha) + tV\alpha\rho] = 0$, or $V\rho\psi\rho - t[V\alpha\psi\rho + V\rho\psi\alpha - \rho V\alpha\rho] + t^2(V\alpha\psi\alpha - \alpha V\alpha\rho) = 0$. So $SV\rho\psi\rho\, V(\alpha\psi\rho + \rho\psi\alpha - \rho V\alpha\rho)\, V(\alpha\psi\alpha - \alpha V\alpha\rho) = 0$ is the equation required. This exhibits the existence of the triple line, since the origin, which is an arbitrary point on the line, is triple.

Again, eliminating t in the first instance instead of g_1, since

$$(\psi - g_1)\rho + t[V\alpha\rho - (\psi - g_1)\alpha] = 0, \quad V.\ V\alpha\rho(\psi - g_1)\rho = V(\psi - g_1)\,\alpha(\psi - g_1)\rho.$$

If $m_3 = M_3 - S\epsilon\Psi\epsilon$ is the third invariant of ψ, this becomes

$$V\ V\alpha\rho(\psi - g_1)\rho = m(g_1)(\psi' - g_1)^{-1}\ V\alpha\rho = m_3\psi'^{-1}V\alpha\rho + g_1\psi' V\alpha\rho + g_1^2 V\alpha\rho,$$

and $S(m_3\psi'^{-1}\ V\alpha\rho + V\psi\rho\ V\alpha\rho)(\psi' V\alpha\rho - \rho V\alpha\rho)\ V\alpha\rho = 0$ is a second form of the equation of the quintic also exhibiting the triple line.

The cone of infinite directions of this surface is $V^2\alpha\rho\, S\alpha\rho\, V\rho\psi\rho = 0$, consisting of the director and an imaginary quadric cone.

If the triple line $\epsilon + t\alpha$ is a line of the congruency $\psi\alpha = a\alpha$, and the first equation of the quintic reduces to $SV\rho\psi\rho\ V[\alpha(\psi - a)\rho - \rho V\alpha\rho]V\alpha V\alpha\rho = 0$. The line is quadruple, and the single real sheet is limited by the extreme points on the line.

If, however, $\psi'\alpha = a\alpha$, so that $\epsilon + t\alpha$ is a line of the reciprocal congruency, $(\psi + tV\alpha - g_1)(\rho - \epsilon - t\alpha) = 0$ becomes

$$(\psi + tV\alpha - g_1)(\rho - \epsilon) = t(a - g_1)\alpha + 2tV\epsilon\alpha.$$

Operating on this by $S\alpha^{-1}$, $(a - g_1)[S\alpha^{-1}(\rho - \epsilon) - t] = 0$. If $g_1 = a$, the vector equation is $(\psi - a)(\rho - \epsilon) + tV\alpha(\rho + \epsilon) = 0$. Hence $S(\rho + \epsilon)\, t\, (\psi - a)(\rho - \epsilon) = 0$, which reduces at once to the root quadric $S\rho(\Psi - a)\rho = M(a)$.

But if $t = S\alpha^{-1}(\rho - \epsilon)$, operating on the vector relation by $SV\alpha(\rho - \epsilon)$,

$$\begin{aligned} S\alpha^{-1}(\rho - \epsilon)\ V^2\alpha(\rho - \epsilon) &= -S\alpha(\rho - \epsilon)\,\psi(\rho - \epsilon - t\alpha) \\ &= +S(\rho - \epsilon - t\alpha)\ V[\alpha\psi'(\rho - \epsilon) + \psi'\alpha(\rho - \epsilon)] \\ &= -S\alpha V(\rho - \epsilon)\psi'(\rho - \epsilon), \end{aligned}$$

using the identical relation $\psi V\lambda\mu + V\psi'\lambda\mu + V\lambda\psi'\mu = 0$. This is the equa-

tion of a cylindroid, illustrating a general law governing the distribution of the lines of shortest distance between consecutive lines of a congruency, and due, I believe, to Sir W. R. Hamilton.

Taking the origin at $-\alpha' = -V\alpha^{-1}\Psi\alpha$ the foot of the perpendicular on the line, the equation of the cylindroid $S\alpha^{-1}\rho V^2\alpha\rho + S\alpha\rho\psi'\rho = 0$ may be referred, in Cartesians, to the lines β and γ through $-\alpha'$ given in Art. 9.

$$U\beta U\gamma = U\alpha, \text{ if } b > c, \text{ as } \beta = \frac{(\Psi - c)\alpha'}{b-c}, \quad \gamma = \frac{(\Psi - b)\alpha'}{c-b}, \quad \text{and} \quad \beta\gamma = \alpha \cdot \frac{T^2\alpha'}{b-c}.$$

Now ψ is $\Psi - V\alpha'$ and $S\alpha\rho\psi'\rho = S\alpha\rho\psi'[\alpha^{-1}S\alpha\rho + \alpha^{-1}V\alpha\rho] = S\alpha\rho[\psi V\alpha\rho \cdot \alpha^{-1} + V\alpha\rho\psi\alpha^{-1}] = S\alpha^{-1}V\alpha\rho\,\psi V\alpha\rho$. If $\rho = xU\alpha + yU\beta + zU\gamma$, $VU\alpha\rho = yU\gamma - zU\beta$, and the cylindroid is $x(y^2+z^2) = (b-c)yz$.

The axes i, j, k of Ψ are lines of both congruencies. Hence the locus of axes of $\Psi\rho + xVi\rho = g\rho$ is $x(y^2+z^2) = (G_2 - G_3)yz$. Of course there is nothing indefinite about the cylindroid, but, if the axes at the double point on the focal surface are directly investigated, indefiniteness occurs. G_1 is the double root, and $\Psi\rho + r_1 Vi\rho = G_1\rho$ is satisfied by i and $j\sqrt{G_3 - G_1} - k\sqrt{G_1 - G_2}$, and therefore by any vector coplanar with these, r_1 being $\sqrt{(G_3 - G_1)(G_1 - G_2)}$.

It has been proved in Art. 4 that lines of the congruency parallel to a given plane meet the line of the adjoint congruency normal to the plane. They therefore lie on a cylindroid. The finite lines of the quintics whose triple line is at infinity generate a cylindroid.

The equation of a cylindroid may be thrown into a form analogous to that of the nodal surface. If ρ is the vector to any point on a line $V\rho_1^{-1}\Psi\rho_1 + y\rho_1$ at right angles to α, and if $\psi = \Psi + V\rho$, $(\psi' - g_2)(\psi' - g_3)\alpha = 0$, as α is coplanar with two axes of ψ'. So $S\alpha\psi'.\alpha\psi'^2\alpha = 0$. The pedal surface has for its equation $S\rho\psi'\rho\psi'^2\rho = 0$, and the nodal surface has $S\rho\psi\rho\psi^2\rho = 0$ for its equation. If α is variable, and ρ fixed, $S\alpha\psi'\alpha\psi'^2\alpha = 0$ is the equation of the triad of planes through pairs of the axes of ψ'.

17. If $\psi\alpha = a\alpha$, the cylindroid of lines of the first congruency through $-\alpha' + y\alpha$ is, when referred to the centre of the congruency,

$$S\alpha(\rho + \alpha')\psi(\rho + \alpha') + S\alpha^{-1}\rho V^2\alpha(\rho + \alpha') = 0.$$

The cylindroid of lines of the second congruency through $+\alpha' + y\alpha$ is

$S\alpha(\rho-\alpha')\psi(\rho-\alpha')-S\alpha^{-1}\rho V^2\alpha(\rho-\alpha')=0$, as follows by change of sign of Ψ, and therefore of $\alpha'=V\alpha^{-1}\Psi\alpha$. In vectors, by the aid of the Cartesian equation, the cylindroids are

$$\rho=-\alpha'+\tfrac{1}{2}(b-c)U\alpha\sin 2\theta+t(U\beta\cos\theta+U\gamma\sin\theta),$$
$$\rho=+\alpha'-\tfrac{1}{2}(b-c)U\alpha\sin 2\theta'+t(U\beta\cos\theta'+U\gamma\sin\theta').$$

Hence, for lines in the same plane, normal to α, $\sin 2\theta+\sin 2\theta'=0$; or $\theta'+\theta=\pi$, or $\theta'-\theta=\frac{\pi}{2}$, as it is convenient to suppose the angles positive. Also, the lines of each congruency are equally inclined to β and γ.

If AC and AD are the lines of the first, and BC and BD those of the second, their roots are g_1 and g_2, respectively (Art. 10). The bisectors

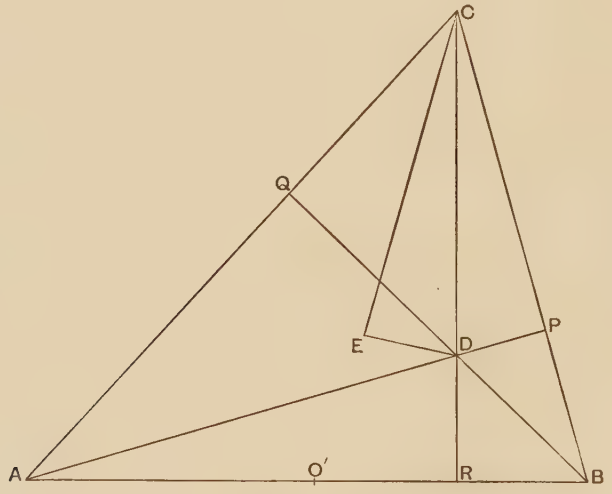

FIG. 7.

of the angles at C and D are parallel to β and γ. The bisector O' of AB is the foot of the central perpendicular on the plane. (See fig. 7.)

Lines of the congruencies pass through A, B, P and Q, whose roots are equal to α. A right circular cylinder, whose axis is parallel to α, and of which AB is a diameter, passes though P and Q, since the angles at these points are right. Hence the locus of P and Q is a conic, the cylinder and root quadric defined by a, having as common generators

the axes of the cylindroids. A cylindroid has two generators of root α, so the intersection of a cylindroid and the root quadric consists of the conic, the two generators, and the axis of the cylindroid counted twice.

If ρ is the vector from the centre to C, $(\Psi + V\rho - g_1)\rho_1 = 0$, and $(\Psi - V\rho - g_1)\rho'_1 = 0$. Hence, as $V\rho_1\rho'_1$ is parallel to $(\Psi - g_1)\rho$ and also to α,

$$\rho = \tfrac{1}{2}(b-c)\sin 2\theta \frac{(\Psi-g_1)^{-1}U\alpha}{S\alpha^{-1}(\Psi-g_1)^{-1}\alpha}, \text{ since } T\alpha S\alpha^{-1}\rho = \tfrac{1}{2}(b-c)\sin 2\theta,$$

$$\text{or } \rho = \sqrt{(b-g_1)(g_1-c)}\,\frac{(\Psi-g_1)^{-1}U\alpha}{S\alpha^{-1}(\Psi-g_1)^{-1}\alpha}, \text{ since } g_1 = b\cos^2\theta + c\sin^2\theta,$$

a quartic curve lying on the cone $S\alpha\rho\Psi\rho = 0$. This quartic, the conic, and the cubic $S\alpha^{-1}\rho V^2\alpha\rho = 0$ in the plane infinity, are common to the cylindroids.

If AB makes an angle θ_0 with ED, the direction of β,

$$AQ = AB\cos(\theta - \theta_0).$$

Therefore, the vector to Q is

$$\begin{aligned}\rho = &-\alpha' + \tfrac{1}{2}(b-c)\sin 2\theta\,\alpha \\ &+ [U\alpha'\cos(\theta-\theta_0) + U(\alpha\alpha')\sin(\theta-\theta_0)]\,2T\alpha'\cos(\theta-\theta_0) \\ &= \tfrac{1}{2}(b-c)\sin 2\theta\,\alpha + \alpha'\cos 2(\theta-\theta_0) + U\alpha\cdot\alpha'\sin 2(\theta-\theta_0).\end{aligned}$$

The middle point of AQ is the foot of the central perpendicular on the line AQ, and the locus of these points is

$$\rho = -\tfrac{1}{2}\alpha' + \tfrac{1}{2}(b-c)\sin 2\theta\cdot\alpha + \tfrac{1}{2}\alpha'\cos 2(\theta-\theta_0) + \tfrac{1}{2}U\alpha\cdot\alpha'\sin 2(\theta-\theta_0),$$

a conic lying on a cylinder parallel to α, of which AO' is a diameter. When $\theta = \frac{\pi}{4}$, both pairs of lines coincide, and A, B, and the coincident points P, Q, C, and C' are on the focal surface. The point of contact of the plane with the surface is the point of intersection of the generators, or $\rho = \tfrac{1}{2}(b-c)\alpha + (U\beta\sin\theta_0 + U\gamma\cos\theta_0)T\alpha'$. The angle θ_0 may be replaced by the aid of the relations in Art. 9, for

$$\tan\theta_0 = -\frac{TV\beta\alpha'}{S\beta\alpha'} = \frac{T\alpha}{c-\alpha'} = \sqrt{\frac{a'-b}{c-a'}}, \text{ since } T^2\alpha = (a'-b)(c-a').$$

18. Through any point P on the nodal ellipse of a quintic pass two generators meeting the triple line in A and B. The line PC' at right angles to these, and therefore to the triple line, is a line of the adjoint congruency, and consequently a generator of the cylindroid whose axis is $V\alpha^{-1}\Psi\alpha + t\alpha$, the generator of the quintic parallel to the triple line $\epsilon + t\alpha$. The nodal conic lies, therefore, on this cylindroid. Again, the planes through P and the axes of the cylindroid and the quintic cut at right angles, the normal to the first plane being at right angles to PC' and the axis of the cylindroid, and the normal to the second plane being PC'. The nodal ellipse lies on a right circular cylinder parallel to α, containing the axes of the cylindroid and the quintic in a diametral plane. Since any line in the plane PAB is perpendicular to PC', the nodal ellipse is the locus of the feet of perpendiculars on generators of the cylindroid from any point on the axis of the quintic. Every generator of the cylindroid meets five generators of the quintic; three it cuts at right angles, and its root is equal to that of the remaining two. The three generators consist of the axis of the cylindroid and the two generators through the point in which the generator of the cylindroid meets the nodal ellipse.

The plane of the ellipse contains a generator of each surface, $Q'P'R$ of the cylindroid and QPR of the quintic. $Q'R$ meets the ellipse in Q' on the axis, and in P'; at each of these points it meets a pair of generators of the quintic, and it must meet the fifth QR in a point not on the conic. These lines do not cut at right angles, for the generator of the cylindroid through P is at right angles to QR. Consequently, the roots of QR and $Q'R'$ are equal.

The tangent plane to the right circular cylinder through the axis of the quintic contains a pair of generators of the quintic intersecting at Q. The generator of the cylindroid through Q has its root equal to that of QR or $Q'R$, and the second generator of the quintic through Q' has its root also equal to this. The centre of the ellipse must lie in the plane through the centre of the cylindroid at right angles to the axis. The vector to the centre is therefore $\frac{1}{2}(V\alpha^{-1}\Psi\alpha + V\alpha^{-1}V\alpha\epsilon)$, or $\frac{1}{2}V\alpha^{-1}\psi'\alpha$.

Using the vectorial equation of the cylindroid, $\rho = \alpha' - \frac{1}{2}(b-c)U\alpha\sin 2\theta + y(U\beta\cos\theta + U\gamma\sin\theta)$ as given in the last Article, the normal at y, θ is

parallel to $V\frac{d\rho}{dy}\frac{d\rho}{d\theta}$, or to $\nu = (b-c)\cos 2\theta\,(U\beta\sin\theta - U\gamma\cos\theta) - yU\alpha$, and the equation of the tangent plane at y', θ' is $S(\rho-\alpha')\nu' + \frac{1}{2}y'(b-c)\sin 2\theta' = 0$. Substituting in this the equation of an arbitrary generator, $y\cos 2\theta'\sin(\theta-\theta') = y'\sin(\theta-\theta')\cos(\theta+\theta')$ is the result. The equation of the conic section in the plane is therefore

$$\rho = \alpha' - \tfrac{1}{2}(b-c)\,U\alpha\sin 2\theta + y'\frac{\cos(\theta+\theta')}{\cos 2\theta'}(U\beta\cos\theta + U\gamma\sin\theta),$$

or, $\rho = \alpha' + \frac{1}{2}y'\sec 2\theta'\,(U\beta\cos\theta' - U\gamma\sin\theta')$,

$$-\tfrac{1}{2}[(b-c)\,U\alpha + y'\sec 2\theta'\,(U\beta\sin\theta' - U\gamma\cos\theta')]\sin 2\theta,$$

$$+\tfrac{1}{2}y'\sec 2\theta'\,(U\beta\cos\theta' + U\gamma\sin\theta')\cos 2\theta$$

referred to its principal axes. The axis minor is parallel to the generator in the plane of the conic. The vector to Q' is $\alpha' - \frac{1}{2}(b-c)\sin 2\theta' U\alpha$, and that to Q is $\alpha' + y'\sec 2\theta'(U\beta\cos\theta' - U\gamma\sin\theta') + \frac{1}{2}(b-c)\sin 2\theta' U\alpha$, since QQ' is a diameter of the ellipse. Hence the axis of the quintic is

$$\alpha' + y'\sec 2\theta'\,(U\beta\cos\theta' - U\gamma\sin\theta') + t\alpha.$$

As y' varies, the axes lie in a plane, and β and γ bisect the angles between the trace of this plane on $S\alpha\rho = 0$, and the direction of the generator.

The lines of the congruency in the plane at right angles to α, which passes through $\rho = \frac{1}{2}(b-c)\,U\alpha\cos u$ cut at an angle u. But $\alpha' + \frac{1}{2}(b-c)\,U\alpha$ is on the focal surface, the double root being $-\frac{1}{2}a$ (Art. 4), so

$$\alpha'^2 + \tfrac{1}{4}(b-c)^2(U\alpha)^2 = M_2 + \tfrac{3}{4}a^2,$$

or in terms of ρ the central perpendicular on the plane, $V^2\rho^{-1}\Psi\rho + \rho^2\sec^2 u = M_2 + \frac{3}{4}S^2\rho^{-1}\Psi\rho$, or thus, $\rho^2(M_3 S\rho\Psi^{-1}\rho - \rho^4\sec^2 u) = \frac{1}{4}S^2\rho\Psi\rho$. This is the locus of the feet of the central perpendiculars on planes containing two generators of the congruency cutting at a given angle. For $u = 0$, the surface is the inverse of the reciprocal of the focal surface. The tangential equation of the surface enveloped by planes $S\lambda\rho + 1 = 0$, containing generators cutting at the angle u, follows on replacing ρ by $-\lambda^{-1}$, and is

$\lambda^2(\cos^2 u\, M_3 S\lambda\Psi^{-1}\lambda - 1) = \frac{1}{4}\cos^2 u\, S^2\lambda\Psi\lambda$. Noting that $\cos u$ occurs in conjunction with Ψ, the pointal equation of this surface is

$$4(M_2\cos^2 u - \rho^2)^3 + 27\cos^2 u\,(M_3\cos^2 u - S\rho\Psi\rho)^2 = 0,$$

– the focal surface of the congruency $\rho = \cos u\, V\alpha^{-1}\Psi\alpha + y\alpha$. These surfaces are similar to the original focal surface, and are obtained from it by diminishing each central radius in the ratio $\cos u$ to unity. When $u = \frac{\pi}{2}$, the surface shrinks to a point.

The locus of the points of intersection of lines of the congruency which cut at the angle u is to be found by elimination of α from $\rho = -V\alpha^{-1}\Psi\alpha + \frac{1}{2}(b-c)\,U\alpha\cos u$. For different values of u a series of surfaces is obtained intermediate between the focal surface $u = 0$, and the nodal surface $u = \frac{\pi}{2}$.

19. In Art. 10 the vector equation of the nodal conic of the quintic whose triple line is $\epsilon + y\alpha$ is given as $\rho = -m(t)\dfrac{(\psi - t)^{-1}\alpha}{\alpha'^{-2}\left[(\psi' - t)\alpha'\right]^2}$ where $\alpha' = V\alpha^{-1}\,\psi\alpha$ and α satisfies $S\alpha\psi\alpha\psi^2\alpha = V^2\alpha\psi\alpha$. Now $V\alpha'^{-1}\psi'\alpha' = m_3\dfrac{V.\ V\alpha\psi\alpha\ V\psi^{-1}\alpha\alpha}{V^2\alpha\psi\alpha} = \alpha$, and α' satisfies $S\alpha'\psi'\alpha'\psi'^2\alpha' = V^2\alpha'\psi'\alpha'$. Again, if $\varpi = V\rho^{-1}\,\psi\rho = \dfrac{V(\psi - t)^{-1}\alpha\psi(\psi - t)^{-1}\alpha}{\left[(\psi - t)^{-1}\alpha\right]^2}$

$= \dfrac{(\psi' - t)\,\alpha'}{m(t)\,\alpha^{-2}\left[(\psi - t)^{-1}\alpha\right]^2}$, $V\varpi^{-1}\psi'\varpi = -\rho$, since $S\rho\psi\rho\psi^2\rho + V^2\rho\psi\rho = 0$, and as $\varpi = V\rho^{-1}\psi\rho = \dfrac{V.\,V\varpi\psi'\varpi\psi\,V\varpi\psi'\varpi}{V^2\varpi\psi'\varpi} = \varpi\cdot\dfrac{S\varpi\psi'\varpi\psi'^2\varpi}{V^2\varpi\psi'\varpi}$, ϖ satisfies $S\varpi\psi'\varpi\psi'^2\varpi = V^2\varpi\psi'\varpi$. A one-to-one relation exists consequently between ϖ and ρ, and as ρ moves along the conic, ϖ traces out the plane section of the quartic surface satisfied by it which is at right angles to α.

The equation of the line $\rho + y\varpi$ referred to the centre of the congruency is $\rho + \epsilon + y\varpi$ or $-V\varpi^{-1}\psi'\varpi + \epsilon + y\varpi$ or $-V\varpi^{-1}\Psi\varpi + y\varpi$, and these lines are the generators of the cylindroid whose axis is $V\alpha^{-1}\Psi\alpha + y\alpha$. Again, the line $\varpi + y\rho$ is $\epsilon + V\rho^{-1}\psi\rho + y\rho$ or $V\rho^{-1}\Psi\rho + y\rho$ when referred to the centre of the congruency.

The cross connexions between the two surfaces $S\rho\psi\rho\psi^2\rho = V^2\rho\psi\rho$ and $S\rho\psi'\rho\psi'^2\rho = V^2\rho\psi'\rho$ may be symmetrically stated as follows :—If $-\beta$ is

any point on the conic of $\epsilon + y\alpha$, $\beta = m(t)\,\frac{(\psi - t)^{-1}\alpha}{\alpha'^2\,[(\psi' - t)\,\alpha']^2}$, and if $\beta' = V\beta^{-1}\psi\beta$

$\beta' = \frac{(\psi' - t)\,\alpha'}{m(t)\,\alpha^{-2}\,[(\psi - t)^{-1}\alpha]^2}$. Now α is parallel to $(\psi - t)\,\beta$, so the relation between α' and β' is $\alpha' = m(t)\,\frac{(\psi' - t)^{-1}\beta'}{\beta^{-2}\,[(\psi - t)\,\beta]^2}$, and that between β and α is

$\alpha = \frac{(\psi - t)\,\beta}{m\,(t)\,\beta'^{-2}\,[(\psi' - t)^{-1}\,\beta']^2}$. Hence if $-\beta$ is any point on the conic of $\epsilon + y\alpha$, $-\alpha'$ is a point on the conic of $-\epsilon + y\beta'$; also β' is a point on the plane section of the second quartic surface made by the plane through the triple point at right angles to α, and α is a point on the plane section of the first quartic made by a similar plane at right angles to β'. Or the conics of lines through ϵ determined by points (α) on the plane section of the first surface at right angles to β', pass through a fixed point $(\epsilon - \beta)$.

Again, taking any two points β and β_1 on the first surface, the normal to the central plane is $V\beta\beta_1$ or α'. The vector $\alpha = V\cdot(V\beta\beta_1)^{-1}\,\psi' V\beta\beta_1$ is uniquely determined, provided β and β_1 are actual. Hence the conic of $\epsilon + y\alpha$ is known when two arbitrary points on it are assigned. Now the line joining β and β_1 meets the quartic surface again in β_2 and β_3, so the three tangent planes which can be drawn to touch the surface through this line separate these four points into three arrangement of pairs. In one tangent plane, one conic passes through β and β_1, and the other through β_2 and β_3, and no conic in either of the other tangent planes passes through β and β_1, or β_2 and β_3. The only case of exception appears to be when β or β_1 is an axis of ψ. The axes of ψ are double lines on the first quartic and the two conics in any tangent plane have a point on each axis common. A conic cannot be uniquely determined by a point on an axis of ψ and an arbitrary point.

20. An arbitrary plane section of the quintic surface has two double points on the nodal ellipse, and a triple point on the axis. Its deficiency is therefore unity, and the vector equation of the section may be expressed by means of elliptic functions. The arbitrary section can have at most three real asymptotes, since three generators of the quintic are parallel to a given plane.

The section in a tangent plane consists of a generator and a quartic. The quartic has two double points, one on the central axis or triple line, and one on the nodal conic. It can have but two real asymptotes, for the generator in its plane passes through a point on the quartic on the nodal conic, and through the double point on the triple line. It cannot, therefore, touch the quartic elsewhere.

The planes containing two generators intersecting on the triple line, cut the surface in cubic curves with one real asymptote. It is not difficult to investigate these cubics analytically. Taking the first form of the equation of the quintic given in Art. 16, referred to an arbitrary point on the triple line, if ρ_2 and ρ_3 are two of the axes of ψ, and therefore parallel to two of the generators of the quintic through the origin,

$$V\rho\psi\rho = V(y\rho_2 + z\rho_3)\,\psi(y\rho_2 + z\rho_3) = yz(g_3 - g_2)V\rho_2\rho_3 = yz(g_3 - g_2)\rho'_1,$$

where ρ is the vector to any point in the plane of these generators, and the equations of the cubic are

$$S\rho\rho'_1 = 0, \quad S\rho'_1 V(\alpha\psi\rho + \rho\psi\alpha - \rho V\alpha\rho)V(\alpha\psi\alpha - \alpha V\alpha\rho) = 0.$$

Expanding by the third vector, the equation of the cubic surface, on which the plane cubic lies, becomes

$$S\rho'_1(\psi\alpha - V\alpha\rho)(S\alpha\rho\psi\alpha - V^2\alpha\rho) = S\rho'_1\alpha S\alpha\psi\rho\,(\psi\alpha - V\alpha\rho),$$

and the cubic may be generated by corresponding intersections of the homographic systems

$$S\rho'_1\alpha\rho = (g_1 - u)S\alpha\rho'_1, \quad S\alpha(\psi - u)\rho\,(\psi\alpha - V\alpha\rho) = 0, \text{ as } \psi'\rho'_1 = g_1\rho'_1.$$

Substituting in these equations $\rho = t\alpha + y\varpi$, where $\psi\varpi + tV\alpha\varpi = h\varpi$,

$$yS\rho'_1\alpha\varpi = (g_1 - u)S\alpha\rho'_1, \quad y(h - u)S\alpha\varpi(\psi\alpha - yV\alpha\varpi) = 0$$

determine the points in which an arbitrary generator of the quintic meets the cubic surface. The quadric of the homographic system, defined by u, contains the two generators of the quintic, whose roots are equal to u. The factor $yV^2\alpha\varpi - S\alpha\varpi\psi\alpha = 0$ gives the point in which the generator meets the nodal ellipse; equating $t\alpha + y\varpi = -\dfrac{m(x)(\psi - x)^{-1}\alpha}{\alpha'^{-2}[(\psi' - x)\alpha']^2}$, this is easily

shown. Since $S\rho'_1\psi\varpi = g_1 S\rho'_1\varpi = hS\rho'_1\varpi - tS\rho'_1\alpha\varpi$, and since the point in which the generator meets the plane $S\rho\rho'_1 = 0$ is given by $tS\alpha\rho'_1 + yS\varpi\rho'_1 = 0$, this generator meets the plane of the homographic system, defined by $u = h$, on the plane $S\rho\rho'_1 = 0$. Thus it is proved that the system of lines $S\rho'_1 V\alpha\rho = (g_1 - u)S\alpha\rho'_1$, $S\rho\rho'_1 = 0$, meets the plane cubic in the points in which it is met by generators of the quintic whose roots are equal to u. These lines are parallel to $\rho'_1 V\alpha\rho'_1$. But the asymptotic cone to the cubic surface is parallel to $V^2\alpha\rho S\rho'_1\alpha\rho = 0$, and the section of this cone by $S\rho\rho'_1 = 0$ gives the lines parallel to the asymptotes of the plane cubic. The lines are therefore parallel to the real asymptote of the plane cubic. The equation of the real asymptote may be thus found. By Art. 15, $\rho'_1 V\alpha\rho'_1$ is parallel to a generator of the quintic, so its root u_0 satisfies $u_0 V^2\alpha\rho'_1 = S\alpha\psi(\rho'_1 V\alpha\rho'_1)$, and the plane of the homographic system, defined by u_0, contains the asymptote.

The perpendicular vector from the origin to a line in the plane $S\rho\rho'_1 = 0$, and parallel to the asymptote is $pUV\alpha\rho'_1$. In terms of u, this is

$$pS\rho'_1 V\alpha UV\alpha\rho'_1 = pTV\alpha\rho'_1 = (g_1 - u)S\alpha\rho'_1, \text{ or } p = (u - g_1)\tan\theta,$$

where θ is the angle the plane of the section makes with the triple line. If u_1 and u_2 are two values of u, $p_1 - p_2 = (u_1 - u_2)\tan\theta$, and for two other values $\dfrac{p_1 - p_2}{p_3 - p_4} = \dfrac{u_1 - u_2}{u_3 - u_4}$, a constant relation for all plane cubics on the surface.

Now, if two real generators, $t\alpha + y\varpi$, have their roots equal to u, t must be real if u is real, or the roots of the quadratic in t, $M(u) = S(\epsilon + t\alpha)(\Psi - u)(\epsilon + t\alpha)$ must be real. The condition for this is

$$D = [M(u) - S\epsilon(\Psi - u)\epsilon]\, S\alpha(\Psi - u)\alpha + S^2\epsilon(\Psi - u)\alpha$$

must be positive. If D is positive, the line defined by u, parallel to the real asymptote, meets the plane cubic in real finite points. The tangents parallel to the real asymptote are defined by the roots u of the quartic $D = 0$. $D = M(u)[S\alpha(\Psi - u)\alpha + SV\alpha\epsilon(\Psi - u)^{-1} V\alpha\epsilon]$ is a form of the quartic, which shows that these roots are independent of the choice of origin on the triple line. Hence four fixed generators of the quintic pass through the

points of contact of tangents parallel to the real asymptote of all plane cubics on the surface.

If $D = 0$ has two real and two imaginary roots, all the plane cubics on the surface are unipartite. In Art. 11, this quartic D was examined in connexion with the director cone.

There are two apparent exceptions to the permanence of type of the plane cubics on a quintic—(1) when $S\alpha\rho'_1 = 0$; (2) when $V\alpha\rho'_1 = 0$. In the first case, ρ'_1 is given by $\rho^2 S\rho\psi\alpha = 0$, the reduced equation of the director cone. The real solution is $\rho'_1 = V\alpha^{-1}\psi\alpha$, and the section lies in the plane $S\rho\alpha\psi\alpha = 0$, which contains the triple line and a pair of generators intersecting on the triple line. If $V\alpha\rho'_1 = 0$, the plane is at infinity.

21. The homographic system of quadrics $S\alpha(\psi - u)\rho(\psi\alpha - V\alpha\rho) = 0$ contains the triple line, and the generator $V\alpha^{-1}\psi\alpha + y\alpha$ of the quintic, the nodal ellipse, and two variable generators. Referred to $\frac{1}{2}\alpha' = \frac{1}{2}V\alpha^{-1}\psi\alpha$ as origin, the equation of the system is, if $S\epsilon\alpha = 0$, $SV\alpha(\psi - u)\rho\, V\alpha^{-1}\rho + \frac{1}{4}S\alpha'(\psi - u)\alpha' = 0$, and the quadrics have $\frac{1}{2}\alpha'$ as a common centre. The quadric corresponding to $u = \infty$ is the right circular cylinder $V^2\alpha\rho + \frac{1}{4}\alpha^2\alpha'^2 = 0$.

Substituting the equation of the generator $\rho = t\alpha + y\varpi$ in the cubic surface of the last article, the result is, $yS\rho'_1[(\psi - h)\alpha - yV\alpha\varpi][S\alpha\varpi\psi\alpha - yV^2\alpha\varpi] = 0$. The second factor may be written

$$\frac{1}{t}S\rho'_1(\psi - h)[t\alpha + y\varpi], \text{ or } \frac{g_1 - h}{t}S\rho'_1(t\alpha + y\varpi),$$

and answers generally to the point in which the generator meets the plane cubic. It vanishes also if $\frac{g_1 - h}{t} = 0$, that is, the cubic surface contains the second generator of the quintic whose root is g_1, and for which t is not zero. Without entering into details, it may be stated that the cubic and quintic have as their complete intersection—(1) the triple line counted three times; (2) the line $V\alpha^{-1}\psi\alpha + y\alpha$ counted twice as the surfaces touch along this line; (3) the nodal ellipse counted twice; (4) the generator whose root is g_1, and for which t is not zero; (5) the imaginary conic $V^2\alpha\rho = 0$ at infinity, and (6) the plane cubic.

If $u = \alpha' = S\alpha'^{-1}\Psi\alpha'$ the quadric breaks up into a pair of planes, one of which contains the triple line and the generator parallel to α, and the other contains the nodal ellipse. The equation of the pair of planes may be written $S\rho(\psi' - tV\alpha - \alpha')\ V\alpha\ V\alpha^{-1}\rho = 0$ where t is arbitrary. Using the notation of Art. 10, $\psi'\alpha' + \alpha\alpha' = \alpha'\alpha' = \psi'_1\alpha'$ if $\psi'_1 = \psi' + V\alpha$, the expression $(\psi'_1 - \alpha')V\alpha V\alpha^{-1}\rho = (\psi'_1 - \alpha')V\alpha' V\alpha'^{-1}(V\alpha V\alpha^{-1}\rho) = -(\psi'_1 - \alpha')V\alpha' a\, S\alpha'^{-1}\alpha^{-1}\rho$. So the pair of planes is $S\rho(\psi'_1 - \alpha')(\psi_1 - \alpha)\alpha\, S\rho\alpha\alpha' = 0$ as $\psi_1\alpha + \alpha'\alpha = \alpha\alpha$.

The system of quadrics meets the plane $S\rho\rho'_1 = 0$ in a system of conics through four fixed points—one on the triple line, one on $\alpha' + y\alpha$, and two on the nodal ellipse. The points on the nodal ellipse are determined by $S\rho'_1(\psi - x)^{-1}\alpha = 0$ or $x^2 + xg_1 + g_1^2 + M_2 - \epsilon^2 = 0$; so g_2 and g_3 are the values of x. The system of quadrics is independent of the position of the origin on the triple line, and serves therefore in generating all the plane cubics on the surface.

The quintic may be considered as generated by the homographic systems

$$S\,V\rho\psi\rho\ V\alpha\rho = S(\psi - u)\,\alpha\, V\rho\psi\rho;\ S\alpha(\psi - u)\,\rho(\psi\alpha - V\alpha\rho) = 0.$$

For $u = \infty$ the cubic reduces to $S\alpha\rho\psi\rho = 0$, and the plane infinity. Substituting the equation of a generator in the cubic defined by u,

$$ty(h - u)(y\,V^2\alpha\varpi - S\alpha\varpi\psi\alpha) = 0.$$

$t = 0$ answers to the three generators of the quintic through the origin which lie on the cubic. These, the triple line, the nodal conic, the generators whose roots are equal to u, and the section of $S\,V\rho\psi\rho\,V\alpha\rho = 0$ by the plane infinity make up the complete intersection with the quintic.

The cubic and quadric defined by u touch along the triple line and intersect in the nodal conic and the two generators of root u.

Again, the quintic may be generated by the homographic systems

$$S(\rho - t\alpha)\psi\rho(\psi\alpha - V\alpha\rho) = 0, \quad S\alpha\rho\psi(\rho - t\alpha) + tV^2\alpha\rho = 0.$$

If $\rho = t'\alpha + y\varpi$, these become

$$yh(t' - t)(S\alpha\varpi\psi\alpha - y\,V^2\alpha\varpi) = 0 \quad \text{and} \quad y(t' - t)(S\alpha\varpi\psi\alpha - y\,V^2\alpha\varpi) = 0.$$

In this case the quintic is generated by triads of axes of linear vector functions.

22. The equation of the tangent plane at any point $t\alpha + y\varpi$, on the quintic, is easily found. If the origin be transferred from ϵ to $\epsilon + t\alpha$, the equation of the quintic is

$$S V\rho\psi_1\rho \;\; V(\rho\psi_1\alpha + \alpha\psi_1\rho - \rho V\alpha\rho) \;\; V\alpha(\psi_1\alpha - V\alpha\rho) = 0$$

where $\psi_1 = \psi + tV\alpha$. Now $\psi_1\varpi = h\varpi$, so if the origin be again transferred to $\epsilon + t\alpha + y\varpi$ by writing $\rho + y\varpi$ for ρ, $V\rho\psi_1\rho$ becomes

$$V(\rho + y\varpi)\psi_1(\rho + y\varpi) = V\rho\psi_1\rho + y V\varpi(\psi_1 - h)\rho.$$

Since no term in this is independent of ρ the tangent plane is simply

$$S V\varpi(\psi_1 - h)\rho \;\; V\varpi[(\psi_1 - h)\alpha - y V\alpha\varpi] \;\; V\alpha(\psi_1\alpha - y V\alpha\varpi) = 0,$$

$y\varpi$ replacing ρ in the second and third vectors, and y^2 coming off as a factor. Expanding the vector product of the second and third vectors

$$[(\psi_1 - h)\alpha - y V\alpha\varpi] \;\; S\alpha\varpi[(\psi_1 - h)\alpha - y V\alpha\varpi]$$

is the result. The scalar factor answers to the nodal ellipse at a point on which the equation of the tangent plane vanishes.

The reduced equation of the plane is

$$S\rho(\psi'_1 - h) \;\; V\varpi[(\psi_1 - h)\alpha - y V\alpha\varpi] = 0,$$

or since, in general, $\psi' V\beta\gamma + V\psi\beta\gamma + V\beta\psi\gamma = M_1 V\beta\gamma = 0$,

$$S\rho V\varpi(\psi_1 + 2h)(\psi_1 - h)\alpha + y S\rho(\psi'_1 - h) \;\; V\varpi V\alpha\varpi = 0.$$

If the point $\epsilon + t\alpha$ is on the focal surface, and ϖ is the double axis $(\psi_1 + 2h)(\psi_1 - h)\alpha$ is parallel to ϖ. The first term vanishes, and the equation of the tangent plane is independent of y, and the tangent plane is torsal, touching the surface at every point on the generator. Thus these quintic surfaces have six torsal lines of this kind, and four of the kind mentioned in Art. 12.

23. The reciprocal of the quintic surface is a quintic. Any plane through the central axis α of the quintic contains a pair of generators intersecting on a conic which meets the axis α. Reciprocally, through every point on the line β reciprocal to α, pass two generators of the reciprocal surface. Their plane envelopes a quadric cone which touches β, the tangent plane through β

being the reciprocal of the point of intersection of α and the conic, and one generator passes through the vertex of the cone as one lies in the plane of the conic. Every plane through β contains three generators, and in one portion of the plane two of the generators intersect on β. The locus of intersection of these generators is consequently a quartic curve which meets β, and has two apparent double points. The surface generated by chords of a quartic with two apparent double points which meet a line β, which the quartic also meets, is a quintic. For, take any plane through β, and draw the connectors of the three points on the quartic not on β; then, as the quartic has two apparent double points, β is a double line on the surface, and the section of the plane consists of three generators, and β counted twice. The cone standing on the quartic whose vertex is common to β and the quartic is irrelevant. Reciprocally, the planes containing two generators of the original surface intersecting on α generate a developable of the fourth class, that is, four planes of the developable pass through an arbitrary point. If the point is on a generator of the quintic, two of the planes pass through the generator, or every generator of the quintic is a "line in two planes," or a bitangent to the developable. If the generator is the single axis through a point on α on the focal surface, these two planes coincide, the two other generators through the point on α being coincident. The single generator is therefore a line of the developable, and the quintic and developable have six lines common. Hence, by reciprocation, six generators of the reciprocal surface touch the quartic curve; and since the curve meets β, eight tangents can be drawn through a line to touch the quartic, the tangent at the point common to β and the quartic counting for two. The rank of the quartic is therefore eight. This may be proved in a different way. Six planes of the developable contain a pair of coincident generators. Through six points on the quartic pass a pair of consecutive generators which meet the quartic again in two points which are consecutive. The line joining these points is a generator of the reciprocal surface and a tangent to the quartic.

A slightly different way of looking at the question is to consider the correspondence between plane sections of one surface and tangent-cones to the other. A section of the reciprocal surface has five double points, one

being on β. From an arbitrary point five double tangent planes may be drawn to the quintic; one is the plane through the central axis, and four are planes of the developable. A section of the quintic has a triple point on α, and two double points on the conic. The singular tangent planes to the reciprocal from an arbitrary point are the plane through β and the tangent planes to the quadric cone. The plane of the conic meets the quintic in the conic counted twice and a generator. The tangent cone to the reciprocal surface through the vertex of the quadric cone consists of the cone counted twice and the plane through β and the generator through the vertex.

The developable formed by the tangent planes to the quintic along the nodal conic is of the fifth class, since the quadric cone touches the reciprocal along a curve of the fifth order. The tangent plane through α to the conic belongs to both sheets of this developable. The reciprocal of this plane is the point in which β touches the cone and meets the quartic. The tangent plane through β to the cone touches the quartic, since it contains a pair of generators intersecting on β. The quartic, therefore, touches the cone, and meets it elsewhere in six points. The reciprocals of these are planes of the quartic developable, touching the ellipse, that is, the planes containing the coincident generators, or the torsal planes of the quintic. The generators of the reciprocal surface through one of the six points are consecutive, and are the reciprocals of the coincident generators of the quintic.

The triple line of the quintic touches the developable of the fourth class where it meets the nodal conic, and meets it in the six points on the focal surface.

24. According to Dr. Salmon's method of generating ruled surfaces by a right line meeting three directing curves of degrees m_1, m_2, and m_3, the degree of the ruled surface is $2m_1m_2m_3 - p_1m_1 - p_2m_2 - p_3m_3$, and the orders of multiplicity of the directing curves are $m_2m_3 - p_1$, $m_3m_1 - p_2$, $m_1m_2 - p_3$, respectively, p_1, p_2, and p_3 being the number of points of intersection of the pairs of curves m_2, m_3; m_3, m_1; and m_1, m_2. (See *Three Dimensions*, Art. 467.)

If m_1 is a right line, the order of its multiplicity requires special investigation. For, if cones standing on m_2 and m_3 have a common vertex on

the right line m_1, the cones have m_2m_3 common generators among which the line m_1 is counted a determinate number of times. m_1 is an edge of multiplicity p_3 on the cone on m_2, and of multiplicity p_2 on the cone on m_3. m_1 is therefore counted p_2p_3 times among the m_2m_3 common edges. Again, m_2 and m_3 intersect in p_1 points. Any plane through m_1 and one of these points is an irrelevant locus of lines meeting the three curves. Discarding these planes, $m_2m_3 - p_2p_3 - p_1$ generators of the ruled surface proper pass through an arbitrary point on m_1. This number is therefore the order of its multiplicity.

Taking as directing curves for the quintic the cubic in a plane containing two generators, the nodal ellipse, and the triple line, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$; $p_1 = 2$, $p_2 = 1$, $p_3 = 1$; and, for the degree of the surface, $2\cdot1\cdot2\cdot3 - 2\cdot1 - 1\cdot2 - 1\cdot3 = 5$; so the quintic is by this method generated without extraneous surfaces. The multiplicity of m_1 is $2\cdot3 - 1\cdot1 - 2 = 3$. If an arbitrary plane section is taken instead of the cubic $m_3 = 5$: $p_1 = 4$, $p_2 = 3$, and $p_3 = 1$, and the multiplicity of m_1 is $2\cdot5 - 3\cdot1 - 4 = 3$. The surface is, in this case, also generated without extraneous surfaces.

This method may be extended to the consideration of the Hessian of the quintic. The Hessian is of the 12*th* degree, and a line of multiplicity g, on the quintic, is of multiplicity $4(g-1)$ on the Hessian. The triple line is of the eighth, and the nodal conic of the fourth order of multiplicity on the Hessian. Any plane through the triple line meets the Hessian again in a quartic with a quadruple point—that is, in four right lines. If the triple line, the conic, and a plane section of the Hessian, are chosen as directing curves, $p_1 = 8$, $p_2 = 8$, $p_3 = 1$: $m_1 = 1$, $m_2 = 2$, $m_3 = 12$. The order of multiplicity of the lines follows correctly, and the Hessian is generated without extraneous factors.

The intersection of the quintic and its Hessian consists of the triple line, counted 24 times, of the conic counted 8 times, and of the ten torsal lines each counted twice.

25. It may be of interest to show how linear vector functions may be applied to the theory of screws. If $\beta_1 + x\alpha_1$, $\beta_2 + x\alpha_2$, and $\beta_3 + x\alpha_3$, are any three lines which may be supposed to be the axes of screws of pitches a_1, a_2, and a_3, and if wrenches act along each of these, the resultant force is

$\lambda = t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3$, and the resultant couple referred to the origin as base point is $\mu = t_1(a_1\alpha_1 + V\beta_1\alpha_1) + t_2(a_2\alpha_2 + V\beta_2\alpha_2) + t_3(a_3\alpha_3 + V\beta_3\alpha_3) = \phi\lambda$. Here ϕ is the linear vector function defined by

$$\phi\rho = \Sigma \frac{(a_1\alpha_1 + V\beta_1\alpha_1)S\alpha_2\alpha_3\rho}{S\alpha_1\alpha_2\alpha_3}.$$

Referred to any new base point ρ, the couple is $\mu - V\rho\lambda = (\phi - V\rho)\lambda$; and, if ρ is on the axis of the screw of the resultant wrench, whose pitch is g, $(\phi - V\rho)\lambda = g\lambda$. Hence, $\rho = -V\lambda^{-1}\phi\lambda + y\lambda$ is the axis of the resultant screw, and $g = S\lambda^{-1}\phi\lambda$ is its pitch. The axes of the screws of the resultants of any wrenches acting on the three given screws belong, therefore, to one of the congruencies of lines treated of in the present Paper, and every geometrical relation described in it may be applied to problems in Rational Mechanics.

In order not to interrupt the geometrical reasoning concerning the roots and axes of linear vector functions, I have made no earlier reference to these applications, especially as I believe that many of them present no difficulty. The root quadrics are Sir R. S. Ball's pitch quadrics, and it was owing to the degradation of the quintic into his celebrated cylindroid that I was led to perceive the close connexion between the theory of linear vector functions and the theory of screws.

If $\phi = \Phi + V\epsilon$ where Φ is self-conjugate $[\Phi - V(\rho - \epsilon)]\lambda = g\lambda$, and so ϵ is the centre of the congruency. Since the conjugate of ϕ is

$$\phi'\rho = \Sigma \frac{V\alpha_2\alpha_3 S(a_1\alpha_1 + V\beta_1\alpha_1)\rho}{S\alpha_1\alpha_2\alpha_3}, \quad 2\epsilon = \frac{\Sigma a_1 V\alpha_1 V\alpha_2\alpha_3 + \Sigma VV\beta_1\alpha_1 V\alpha_2\alpha_3}{S\alpha_1\alpha_2\alpha_3}.$$

If there are only two fixed screws, the force $\lambda = t_1\alpha_1 + t_2\alpha_2$ is parallel to a fixed plane. The relation $(\phi - V\rho)\lambda = g\lambda$ holds as before if ϕ is a special function, λ being parallel to a fixed plane the axes of resultant screws lie on a cylindroid.

If four arbitrary screws are assigned, the fourth axis being $\beta_4 + x\alpha_4$, the resultant force is $\lambda = t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + t_4\alpha_4$. From a given value of the resultant force the magnitudes of the wrenches cannot be determined unless t_4, say, is given. Suppose $\lambda = \lambda_0 + t_4\alpha_4$, and ϕ to be as before the linear

vector function defined by the three screws, $\mu = \phi\lambda_0 + t_4(a_4\alpha_4 + V\beta_4\alpha_4)$, or $\mu = \phi\lambda - t_4(\phi a_4 - V\beta_4\alpha_4 - a_4\alpha_4)$. If ρ is on the axis of the screw of the resultant wrench, and g is the pitch, $(\phi - V\rho - g)\lambda = t_4(\phi a_4 - V\beta_4\alpha_4 - a_4\alpha_4)$ $= t_4\gamma$, suppose, γ being a fixed vector. This vector equation involves two independent parameters g and t_4, and is the equation of a complex. It easily takes any of the four forms

$$\rho = -V\lambda^{-1}\phi\lambda + t_4 V\lambda^{-1}\gamma + x\lambda; \quad V\gamma(\phi - V\rho - g)\lambda = 0; \quad V\lambda(\phi - V\rho)\lambda = t_4 V\lambda\gamma;$$

and $S\gamma\lambda(\phi - V\rho)\lambda = 0$. If ρ_0 is any second point on a line of the complex through ρ, λ is parallel to $\rho - \rho_0$, so from the fourth of these equations $S\gamma(\rho - \rho_0)(\phi + V\rho_0)(\rho - \rho_0) = 0$, and the lines of the complex through any fixed point ρ_0 lie on a quadric cone. Or writing $\alpha = \rho - \rho_0$, $\beta = V\rho\rho_0$ the complex is $S\gamma\alpha(\phi\alpha - \beta) = 0$, and is of the second order.

For five arbitrary screws and with the same function ϕ,

$$\lambda = \lambda_0 + t_4\alpha_4 + t_5\alpha_5, \quad \mu = \phi\lambda_0 + t_4(a_4\alpha_4 + V\beta_4\alpha_4) + t_5(a_5\alpha_5 + V\beta_5\alpha_5),$$

or $\mu = \phi\lambda - t_4\gamma - t_5\gamma'$. If ρ is on the axis of the resultant screw, and if g is its pitch $(\phi - V\rho - g)\lambda = t_4\gamma + t_5\gamma'$, and the axis is a line of one of the system of complexes $S(t_4\gamma + t_5\gamma')\alpha(\phi\alpha + \beta) = 0$, that is to say, its position is arbitrary. Its pitch however is determinate, for let any arbitrary values consistent with $S\alpha\beta = 0$ be assigned to α and β, $S(t_4\gamma + t_5\gamma')\alpha(\phi\alpha + \beta) = 0$ determines the ratio of t_4 to t_5. So $g = S\alpha^{-1}\phi\alpha - S\alpha^{-1}(t_4\gamma + t_5\gamma')$ – a perfectly definite value, since the tensor of α is proportional to that of $t_4\gamma + t_5\gamma'$.